The structure of cities

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The economic determinants of city structure

Why do we have cities in the first place ? Because of *positive externalities*, which create centripetal forces:

- craftsmen learn from other craftsmen

But there are also *negative externalities*, which create centrifugal forces:

The city comes to existence because the former are stronger than the latter. But then these forces determine the structure of the city. There are two possible ways to allot land:

The two processes will yield *different* structures

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- by a benevolent planner

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The market

The market for land

- the shape of the city is prescribed as some $\Omega \subset R^d$

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- there is a single good in the economy, which all firms produce and all residents consume
- rents and surplus production leave the city

Equilibrium conditions: residents

All inhabitants are identical. Their utility is $U(c, S)$, where c is consumption and S the area they rent. U satisfies the usual conditions. All people who live at x pay the same rent $Q(x)$ and have the same take-home pay $\varphi(x)$ (salary net of transportation costs), so they all solve the same problem:

$$\max_{c,S} \{U(c,S) \mid c + Q(x) S \leq \varphi\}$$

The optimal value $\bar{u}$ must be the same for all x:

$$\bar{u} := \max_{c,S} \{U(c,S) \mid c + Q(x) S \leq \varphi(x)\}$$

$\bar{u}$ is given exogeneously (income earned outside the city). This determines $N(\varphi)$, relative density of residents, and $Q(\varphi)$

Equilibrium conditions: firms

All firms are identical and have constant returns to scale.. The production per unit of land at y is $f(z, n(y))$, where $n(y)$ is the relative density of jobs. The *productivity* $z = z(y)$ is given by:

$$z(y) = g(\int \rho(y', y)\nu(y')dy')$$

where $\rho \geq 0$, g is increasing and $\nu(y)$ is the *absolute* density of jobs. All firms located at y pay the same rent $q(y)$ and the same salary $\psi(y)$, and make the same profit $f(z, n) - \psi n - q(y)$, which is zero because of perfect competition:

$$0 = \max_{n \geq 0} \{f(z, n) - \psi(y)\, n - q(y)\}$$

This gives $n(\varphi)$, relative density of jobs

Transportation costs

Workers who live at x and work at y, where they pick up a salary $\psi(y)$, have a revenue

$$\varphi(x) = \psi(y) - c(x, y)$$

with $c \geq 0$ and $c(x, x) = 0$. It may be convex or concave - the first case is typical of the structure of cities, the second of interregional trade.
A *transport plan* is a map $x \longrightarrow P_x$, where P_x is a probability on Ω. We understand $P_x(y)$ as the proportion of the residents of x who work at y.
This becomes a transport map $\tau : \Omega \longrightarrow \Omega$ if $P_x = \delta_{\tau(x)}$ is a Dirac mass, ie if all those who live at x work at the same place $\tau(x)$. This is the case when the cost is convex.

Equilibrum: definition

- We start from two densities $\mu(x)$ (residents) et $\nu(y)$ (jobs) and two functions $\varphi(x)$ (take-home pay at x) et $\psi(y)$ (salary at y) such that:

$$
\begin{aligned}
\int \mu &= \int \nu \\
\varphi(x) &= \max_y \{\psi(y) - c(x, y)\} \\
\psi(y) &= \min_x \{\varphi(x) + c(x, y)\}
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- We deduce the productivity $z(y)$, and the relative densities $N = N(\varphi(x))$ and $\nu = \nu(z(y), \psi(y))$ as well as the rents $Q = Q(\varphi(x))$ and $q = q(z(y), \psi(y))$ for residences and firms

Equilibrium: definition

- In view of $q(y)$ and $Q(x)$, the landlords allot the land. If $\theta(x)$ is the proportion of land for industrial use, we must have:

$$\theta = \begin{cases} 0 & \text{si } q(z,\psi) - Q(\varphi) < 0 \\ 0 \leq \theta(x) \leq 1 & \text{si } q(z,\psi) - Q(\varphi) = 0 \\ 1 & \text{si } q(z,\psi) - Q(\varphi) > 0 \end{cases}$$

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- We deduce new distributions $\tilde{\mu}$ and $\tilde{\nu}$ by:

$$\begin{aligned} \tilde{\mu}(x) &= (1 - \theta(z,\psi,\varphi)) N(\varphi(x)) \\ \tilde{\nu}(y) &= \theta(z,\psi,\varphi) n(z,\psi) \end{aligned}$$

and there is a unique pair $(\tilde{\varphi}, \tilde{\psi})$ such that $\int \tilde{\mu} = \int \tilde{\nu}$

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- We have an equilibrium if $(\mu, \nu, \varphi, \psi) = (\tilde{\mu}, \tilde{\nu}, \tilde{\varphi}, \tilde{\psi})$

Existence

Theorem

There is always an equilibrium

It can be checked that people move: there is no equilibrium with all residents living where they work.

R. E. Lucas, Jr. and E. Rossi-Hansberg, *On the Internal Structure of Cities*, Econometrica, vol. 70 (2002), pp. 1445-1476. (*Treats the case of a circular city with radial transport and iceberg costs*)

G. Carlier and I. Ekeland, "Equilibrium structure of an bidimensional assymmetric city". *Nonlinear Analysis TWA*, vol 8 (2007): 725 - 748.

The planner

The benevolent president.

The president has a large empty space at his disposal, and assigns locations to firms and workers in order to maximize the social optimum

- The number of workers a given firm employs is exogeneously prescribed (say 1)

$$w(x) \text{ density of residence at } x$$
$$f(x) \text{ density of jobs at } x$$
$$\int_{\mathbb{R}^2} w(x)\,dx = \int_{\mathbb{R}^2} f(x)\,dx = 1$$

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- The number of workers a given firm employs is exogeneously prescribed (say 1)
- Number of firms and workers prescribed exogeneously
- Shape of the city is not prescribed (one- or two-dimensional)

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$$f(x) \text{ density of jobs at } x$$

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Externalities

There are several externalities, some positive, some negative:

- **Congestion**: $\frac{1}{2}\gamma_w w^2(x)$ and $\frac{1}{2}\gamma_f f^2(x)$
- **Transport:** $c(x, y)$
- **Firm-firm** (production) $C_1 - \frac{k}{2} \int \int \|x - y\|^2 f(x) f(y) \, dxdy$
- **Residence-residence** (utility) $C_2 - \frac{\theta}{2} \int \int^2 \|x - y\|^2 w(x) w(y) \, dxdy$
- **Firme-residence** (utility) **:** $C_3 - \rho \int \int^2 \|x - y\|^2 w(x) f(y) \, dxdy$

We can have $\rho > 0$ (convenience) or $\rho < 0$ (pollution).

The planner's problem

It consists of minimizing

$$J(f,w) := \frac{\gamma_w}{2}\int w^2 dx + \frac{\gamma_f}{2}\int f^2 dy + W_c(f,w) + \frac{k}{2}\int\int^2 \|x-y\|^2 f(x) f(
+\frac{\theta}{2}\int\int^2 \|x-y\|^2 w(x) w(y)\, dxdy + \rho\int\int^2 \|x-y\|^2 w(x) f(y)\, dxc$$

over all probability densities (w, f) and all maps T which transport w to f. Here W_c is the Wasserstein distance.

Existence obtains provided centifupetal forces dominate centrifugal ones

$$\gamma_f > 0 \ \ , \gamma_w > 0 \ \ , k > 0, \ \ \theta > 0$$
$$k + \rho > 0, \ \theta + \rho > 0, \ \rho > -\frac{1}{2}$$

The quadratic case

In the case of quadratic cost, there is an explicit solution:

$$T(x) = \lambda x \tag{1}$$

$$w(x) = \frac{1}{\gamma_w}\left[C_w - \left(\theta + \rho + \frac{1-\lambda)}{2}\right)x^2\right]_+ \tag{2}$$

$$f(y) = \frac{1}{\gamma_f}\left[C_f - \left(k + \rho + \frac{1-\lambda^{-1})}{2}\right)y^2\right]_+ \tag{3}$$

So the optimal city is circular, the optimal transport map is radially linear, and the distributions w and f are concentric and bell-shaped. Firms are more concentrated than residences if $\varrho + k > 0$. The size of the city increases with γ and decreases with θ, $\rho + k$, and c
In the case of superquadratic costs, $c(x, y) = \frac{1}{p}\|x-y\|^p$ *the symmetry will break* if congestion costs are relatively small.

The two-sector city

$$C_1 \ : \ = \frac{\gamma_w}{2} \int (w_1 + w_2)^2 + \frac{\gamma_f}{2} \int (f_1 + f_2)^2 \text{ (congestion)}$$

$$C_2 \ : \ = tW_c(f_1, w_1) + tW_c(f_2, w_2) \text{ (transportation)}$$

$$C_3 \ : \ = \sum_{i=1}^{2} \frac{k_i}{2} \int \int |x-y|^2 f_i(x) f_i(y) dxdy \text{ (worker/worker externalities)}$$

$$C_4 \ : \ = \frac{\theta}{4} \int \int |x-y|^2 (w_1(x) + w_2(x))(w_1(y) + w_2(y)) dxdy \text{ (firms/firms}$$

$$C_5 \ : \ = \sum_{i=1}^{2} \frac{\rho_i}{2} \int \int |x-y|^2 f_i(x)(w_1(y) + w_2(y)) dxdy \text{ (firms/workers)}$$

Frame Title

Existence obtains under similar conditions (centripetal parameters should dominate centrifugal ones), and we have a structural result:

Theorem

Assume that $k_1 + \rho_1 \neq k_2 + \rho_2$. Then there is segregation:

$$f_1 f_2 = w_1 w_2 = 0$$

How the planner harnesses market forces

Implementation

To implement the social optimum, the president can impose two land taxes: $B(y)$ for business use and $R(x)$ for land use.
Firms are perfectly competitive, and choose their location so as to maximize profit, ie production minus salary minus business tax. Workers and competitive and choose choose their residence so as to maximize their take-home pay, net of transportation cost and residential tax.
We find (in the quadratic, one-sector case, $\gamma_f = 0$)

$$B(y) = C_4 + \left(\rho + \frac{k}{2}\right) y^2$$

$$R(x) = C_5 - \left[t \frac{\rho + k}{t + \rho + k} + \frac{\theta + \rho}{2}\right] x^2$$

$$s(x) = C_6 - (\rho + k) x^2$$

where $s(x) = C_6 - (\rho + k) x^2$ is the salary at x. Note the counter-intuitive fact that $B(x)$ is an increasing function of the distance to the center

Guillaume Carlier, Ivar Ekeland and Jean-Charles Rochet, "Tax and the city", in preparation