Movements of crowds and transport with an obstacle.

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Outline

1 A model for movement of crowds: G.Buttazzo, C.J., E.Oudet

- A model without congestion
- A model with congestion
- Some questions

2 The transport problem with a convex obstacle in the quadratic case: P. Cardaliaguet, C.J.

- The quadratic MK problem problem with an obstacle
- From the quadratic case to a Monge-like case
- Method
- The transport problem on the boundary
- The optimal transport map

Setting of the problem

- $\Omega \subset R^2$ a compact *possibly non convex* set,
- $f_0 = f_0(x)\ dx$ and $f_1(x)\ dx$ two probability measures: the location of the crowd before and after moving.

Aim

Find the better (the faster way) way for the croud to move from f_0 to f_1 (for instance to escape from a danger).
Unknown: $\rho(x, t)$ the proportion of the crowd at time t and position x where the time has been rescaled to $t \in [0, 1]$.

- In the optimal movement, every one follows the geodesic curve from his starting to his goal point,
- The time lost for one person going from x to y is:

$$d^2(x,y) = \inf_{\gamma \text{ Lipschitz}} \left\{ \int_0^1 |\dot{\gamma}(s)^2| ds \mid \xi(0) = x,\ \xi(1) = y \right\}$$

  where the infimum is taken among every γ such that $\gamma(s) \in \Omega\ \forall s \in [0,1]$.
- In which way we must chose coupling between starting and goal points?

The transport problem

Monge-Kantorovich problem

$$(\overline{MK}) \qquad \inf_{\mu \in \Pi(f_0, f_1)} \left\{ \int_{\Omega \times \Omega} \frac{d^2(x,y)}{2} d\mu(x,y) \right\}$$

where $\Pi(f_0, f_1)$ are the probability measure with fixed marginals f_0 and f_1.

The optimal μ will give the optimal way of rearange the crowd in order to minimize the duration of the movement which is given by $\inf(\overline{MK})$.

The Benamou-Brenier formulations

How can we get $\rho(x,t)$ from μ?

Theorem: J.D.Benamou and Y. Brenier

We have: $\inf(MK) = min(B)$.

$$(B)\ \min \left\{ \int_{\Omega\times[0,1]} \frac{|v(x,t)|^2}{2} d\rho(x,t) \right\}$$

under the constraint:

$$\text{Conservation of mass} \qquad -\partial_t \rho - \mathrm{div}_{x,t}(v\rho) = 0 \text{ inside } \Omega\times]0,1[,$$

$$\text{Initial and final condition} \qquad \rho(\cdot,0) = f_0,\ \rho(\cdot,1) = f_1.$$

Meaning of the constraint

$$\int \partial_t \varphi(x,t)\, d\rho(x,t) + \int \nabla_x \varphi(x,t) \cdot v(x,t)\, d\rho(x,t)$$
$$= \int \varphi(x,1)\, df_1(x) - \int \varphi(x,0)\, df_0(x)$$

for all $\varphi \in \mathcal{C}_c^\infty(R^{d+1})$.

Meaning of $v(x,t)$

$v(x,t)$ is the average velocity of the persons who are at position x at the time t.

Building an optimal transport density $\rho(x,t)$

If $\mu \in \Pi(f_0, f_1)$ is a solution of (MK), the following $(v(x,t), \rho(x,t))$ is optimal for (B):

$$\int \varphi(x,t)\cdot v(x,t)d\rho(x,t) = \int_{\Omega^2}\int_0^1 \varphi(\gamma(t))\cdot \dot{\gamma}_{x_0,x_1}(t)dtd\mu(x_0,x_1),$$

where γ_{x_0,x_1} is a constant speed geodesic joining x_0 to x_1 and $\varphi \in \mathcal{C}_c^\infty(R^{d+1}, R^d)$.

An algorithm by J.D.Benamou and Y. Brenier

We can compute the movement of the optimal crowd without congestion with the algorithm of J.D.Benamou and Y. Brenier.

Images by E. Oudet.

G. Buttazzo, C. J, E. Oudet

Modify (B)

We force ρ to be absolutely continuous

$$\inf \left\{ \int \frac{|v|^2}{2}\rho(x,t) + k\rho^2(x,t)dxdt \right\}$$

under the constraint:

Conservation of mass $\quad -\partial_t \rho - \mathrm{div}_{x,t}(v\rho) = 0$ inside $\Omega \times]0,1[$,

Initial and final condition $\quad \rho(\cdot,0) = f_0, \ \rho(\cdot,1) = f_1.$

Images by E. Oudet

Questions

- Is the optimal ρ unique? (with/without congestion)
- Is the optimal μ unique in the transport model with obstacle?
- Is there an optimal μ not dividing masses in the transport model?

Uniqueness in the model with congestion

The following functionnal is strictly convex in the variable ρ:

$$(v, \rho) \mapsto \int \frac{|v|^2}{2} \rho(x,t) + k\rho^2(x,t) dxdt$$

Non-uniqueness of the optimal transport map

Uniqueness of the optimal transport map and of the optimal ρ does not hold with an obstacle.

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- $E \subset R^d$, compact and convex ,
- $C \subset E$ convex, regular and open: *the obstacle*,
- $f_0 << dx$ and f_1 two probability measures supported on $Int(\Omega)$ with $\Omega := E \backslash C$,

Monge-Kantorovich problem

$$(MK) \qquad \inf_T \left\{ \int_\Omega \frac{d^2(x,Tx)}{2} df_0(x) \right\}$$

where $T : \Omega \to \Omega$ is such that $T^\sharp f_0 = f_1$.

relaxed version:

$$(\overline{MK}) \qquad \inf_{\mu \in \Pi(f_0,f_1)} \left\{ \int_{\Omega\times\Omega} \frac{d^2(x,y)}{2} d\mu(x,y) \right\}$$

where $\Pi(f_0, f_1)$ are the probability measure with fixed marginals f_0 and f_1.
Let μ an optimal transport plan for $(\overline{MK})$.

Theorem

When $f_0 << dx$, it exists an optimal T for (MK).
This means that there exists a solution which does not devide masses.

Linear case

This case has already been studied by M. Felmann and R.J. McCann.
When $f_0 << dx$, it exists an optimal T for (MK).

Usual quadratic case (Y. Brenier 91)

$$(MK) \qquad \inf_{\mu\in\Pi(f_0,f_1)} \left\{ \int_{\Omega\times\Omega} \frac{1}{2}\|x-y\|^2 d\mu(x,y) \right\}$$

Dual formulation:

$$(D) \qquad \sup_{u_1(y)-u_0(x)\leq \frac{\|x-y\|^2}{2}} \left\{ \int_\Omega u_1(y)\ df_1(y) - \int_\Omega u_0(x)\ df_0(x) \right\}$$

Usual quadratic case (Y. Brenier 91)

A triple (μ, u_0, u_1) of solutions of (MK) and (D) satisfies the *primal-dual optimality condition*:
$u_1(y) - u_0(x) = \frac{1}{2}\|x - y\|^2 \; \mu -$ a.e.(x, y).

Idea "with hands"

When we derive the primal-dual optimality condition, we obtain:
$-\nabla u_0(x) = x - y$ that is $y = x + \nabla u_0(x) = Tx$.

Doing the same with an obstacle

Dual formulation:

$$(D) \qquad \sup_{u_1(y)-u_0(x)\leq \frac{d(x,y)^2}{2}} \left\{ \int_\Omega u_1(y)\, df_1(y) - \int_\Omega u_0(x)\, df_0(x) \right\}$$

The triple (μ, u_0, u_1) of solutions of (MK) and (D) satisfies the *primal-dual optimality condition*:
$u_1(y) - u_0(x) = \frac{1}{2}d(x,y)^2 \; \mu -$ a.e.(x,y).
Optimal pair: (u_0, u_1).

Doing the same with an obstacle

When we derive the primal-dual optimality condition, we obtain:

$$-\nabla u_0(x) = \dot{\gamma}_x(0)$$

where γ_x is an optimal constant speed geodesic from x to y.

The starting direction of the movement and the distance are not enough to conclude.

A space-time cost

We introduce the following cost (see P. Bernard and B. Buffoni, C.J.)

$$c((x,s),(y,t)) = \begin{cases} \dfrac{d^2(x,y)}{2(t-s)} & \text{if } s < t \\ 0 & \text{if } \ (x,s) = (y,t) \\ +\infty & \text{otherwise.} \end{cases}$$

$$\inf(MK) = \inf(MK_t)$$

where (MK_t) is the following optimization problem:

$$(MK_t) \quad \inf_{\pi \in \Pi(f_0 \otimes \delta_0, f_1 \otimes \delta_1)} \left\{ \int_{(\Omega \times [0,1])^2} c((x,s),(y,t))\, d\pi((x,s),(y,t)) \right\}.$$

μ optimal for (MK), π optimal for (MK_t):

$$\int \varphi((x,s),(y,t)) d\pi((x,s)(y,t)) = \int \varphi((x,0),(y,1)) d\mu(x,y).$$

Properties

Lemma

$$c((x,s),(y,t)) = \inf\left\{ \int_s^t |\dot\gamma(\tau)|^2 d\tau \;|\gamma(s)=x,\ \gamma(t)=y \right\}$$

where the infimum is taken among the $\gamma : [s,t] \to \Omega$ Lipschitz continuous.

Triangular inequality

Forall $x, y, z \in \Omega$ and $0 \le s < \tau < t \le 1$, we have:

$$c((x,s),(y,t)) \le c((x,s),(z,\tau)) + c((z,\tau),(y,t)).$$

Duality in this New framework

$$(D_t) \qquad \sup_{v \in \mathcal{C}_b(\Omega \times [0,1])} \left\{ \int_\Omega u(y,1)\, df_1(y) - \int_\Omega u(x,0)\, df_0(x) \right\}$$

where the supremum is taken among the u satisfying:
$u(y,t) - u(x,s) \leq c((x,s),(y,t))$.

Theorem: Y. Brenier, C. J, P.Bernard-B. Buffoni, L. Granieri

- $\min(MK_t) = \max(D_t)$,
- (D_t) has a solution u, Lipschitz inside Ω and:

$$\partial_t u(x,s) + \frac{|\nabla_x u(x,s)|^2}{2} = 0 \ \text{ a.e. } (x,s) \in \Omega \times [0,1].$$

- $u(x,0) = u_0(x)$, $u(y,1) = u_1(y)$.

New formulation of the problem

New aim

Find $T(x,s) = (T_x(x,s), T_t(x,s))$ optimal for:

$$\inf_T \left\{ \int_{(\Omega\times[0,1])^2} c((x,s), T(x,s))\, df_0 \otimes \delta_0(x,s) \right\}$$

where the infimum is taken on the T such that:
$T^\sharp f_0 \otimes \delta_0 = f_1 \otimes \delta_1$.
Then $T_x(x,0)$ is a solution for (MK).

Transport set and transport rays

Transport rays

A transport ray is a constant speed geodesic $\gamma : [0,1] \to \Omega$ such that:

$$u_1(\gamma(1)) - u_0(\gamma(0)) = \frac{1}{2} d^2(\gamma(0), \gamma(1)).$$

Transport set

Let Γ be the union of all images of the transport rays.

Method

- Recall the derivate of the *primal dual optimality condition*: if u_0 is derivable at x, $-\nabla u_0(x)$ indicates the starting direction of the movement.
- u_0 is derivable almost everywhere. We reduce to the points where it is derivable.
- To solve the problem, we need to build a map between right parts of the transport rays before the obstacle and after the obstacle.

Simplifying assumption: we assume that all the mass is forced to pass on ∂C during the movement.

Setting of the problem

Let $(x, y) \in \Gamma$ and γ a transport ray joining x to y,
$t_0(x) = \min\{t \geq 0 : \ \gamma(t) \in \partial\Omega\}$,
$t_1(x, y) = \max\{t \geq 0 : \ \gamma(t) \in \partial\Omega\}$,

$$pr_0(x) = (\gamma(t_0(x)), t_0(x)), \ pr_1(x, y) = (\gamma(t_1(x, y)), t_1(x, y)).$$

$$f_0^C := (pr_0)^\sharp f_0, \ f_1^C := (pr_1)^\sharp \mu.$$

Setting of the problem

New aim

Find an optimal transport map for the following problem:

$$(MK_C) \quad \inf_{(T^C)^\sharp f_0^C = f_1^C} \left\{ \int_{\partial\Omega\times[0,1]} c((x,s),T^C(x,s))\, df_1^C(x,s) \right\}.$$

u remains optimal for the dual formulation on the boundary:

$$(D_C) \;\; \sup\{ \int_{\partial C\times[0,1]} u(y,t) df_1^C(y,t) - \int_{\partial C\times[0,1]} u(x,s) df_0^C(x,s)\}$$

under the constraint $u(y,t) - u(x,s) \leq c((x,s),(y,t))$.

Regularity of u

Characterisation of u

$$u(y,t) = \inf_{x\in\Omega}\left\{u_0(x) + \frac{d^2(x,y)}{2t}\right\}$$

- u is derivable inside rays and $\nabla_x u(x,t) = \dot{\gamma}(t)$,
  In particular transport ray cross only tangentially in their inside.
- $\nabla_x u$ has a countable Lipschitz property inside the transport set tangentially to ∂C:

$$\frac{\| < D_x^2 u(x,s)w, w > \|}{\|w\|^2} \leq C \text{ for all } w \text{ tangent to } \partial C \text{ at } x \in \partial C.$$

f_0^C is absolutely continuous

lemma

f_0^C is absolutely continuous with respect to $\mathcal{H}^{d-1}_{\partial C} \otimes dt_{[0,1]}$.

Consider $\theta : (x, s) \mapsto x - s\nabla_x u(x, s)$. It is countably Lipschitz, moreover:

$$\theta \circ pr_0(x) = x.$$

Let $A \subset \partial C \times [0, 1]$ such that $\mathcal{H}^{d-1}_{\partial C} \otimes dt_{[0,1]}(A) = 0$ then:

$$f_0^C(A) = (pr_0)^\sharp f_0(A) = f_0((pr_0)^{-1} A) = f_0(\theta(A)) \leq \mathcal{H}^d(\theta(A))$$

and using the Lipschitz property of θ:

$$\mathcal{H}^d(\theta(A)) \leq \mathrm{Lip}(\theta)\mathcal{H}^d(A) = 0.$$

Solving the problem on the boundary: Different methods

- Transporting along rays, "gluing" the maps together by disintegrating on rays; (S. Sudakov, L.Ambrosio and A.Pratelli, L. Caravenna...)
- Transporting along rays, "gluing" the maps together by making a change of variable to turn rays into horizontal lines and apply Fubini; (M. Feldman and R.J. McCann, L.Caffarelli M. Feldman and R.J. McCann, P. Bernard and B. Buffoni, A. Figalli...)
- Using the method of T. Champion and L. De Pascale;
- other methods: L.C. Evans and W. Gangbo...

Building the optimal transport map

Let $\theta_1(y,t) := y + (1-t)Du(y,t)$. We have:

$$\theta_1(pr_1(x,y)) = y \ \forall (x,y) \in \mathrm{spt}\mu.$$

The optimal transport map

$$T(x) = \theta_1 \circ T^C \circ pr_0(x)$$