Optimal and better transport plans

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Question (Villani):

$$c\text{-cyclical-monotonicity} \Rightarrow \text{optimality?}$$

e.g. for cost function being squared Euclidean distance in $\mathbb{R}^n$.

Answer (Pratelli, S-Teichmann, Beiglböck, Goldstern, Maresch,S.)

Under appropriate assumptions (covering the above special case): YES.

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Under appropriate assumptions (covering the above special case): YES.

Let $(X, \mu), (Y, \nu)$ be polish spaces equipped with Borel probability measures μ, ν and $c : X \times Y \to [0, \infty]$ Borel measurable.
By $\Pi[\mu, \nu]$ we denote the probability measures π on $X \times Y$ with marginals μ and ν.

Definition

For given $c : X \times Y \to [0, \infty]$ a set $\Gamma \subseteq X \times Y$ is called c-cyclically monotone if, for $(x_1, y_1), \ldots, (x_n, y_n) \in$,

$$\sum_{i=l}^{n} c(x_i, y_i) \leq \sum_{i=l}^{n} c(x_i, y_{i+1}),$$

with $y_{n+1} = y_1$.

A measure $\pi \in \Pi(\mu, \nu)$ is called c-cyclically monotone if there is a c-cyclically monotone set Γ with $\pi(\Gamma) = 1$.

Enlightening example (Ambrosio-Pratelli):

$$X = Y = [0,1[\text{ and } \mu = \nu = \text{Lebesgue measure.}$$

For $\alpha \in [0,1[\setminus\mathbb{Q}$ we define $T_\alpha(x) = x + \alpha$, with addition modulo 1. Let

$$c(x,y) = \begin{cases} 1 & \text{if } x = y \\ 2 & \text{if } T_\alpha(x) = y \\ \infty & \text{otherwise} \end{cases}$$

There are two finite transport plans, given by $T_0(x) = x$ and $T_\alpha(x) = x + \alpha$.
Denoting by π_0 and π_α the corresponding measures on $X \times Y$ we have

$$I_c(\pi_0) = \iint_{X \times Y} c \, d\pi_0 = 1$$

$$I_c(\pi_\alpha) = \iint_{X \times Y} c \, d\pi_\alpha = 2$$

Clearly π_0 is the optimal transport plan.

The only finite transport plans are given by the measures $\mu\ \pi_0 + (1-\mu)\ \pi_\alpha$, where $0 \le \mu \le 1$.

There are (modulo null sets) precisely two c-cyclically monotone sets, namely

$$\Gamma_0 = \{(x,x) : x \in [0,1[\ \} \text{ and } \Gamma_\alpha = \{(x, x+\alpha) : x \in [0,1[\}\ .$$

Hence the transport plan π_α is supported by the c-cyclically monotone set Γ_α, but *fails to be optimal*.

Definition (S.-Teichmann):

A transport plan π is called *strongly c-cyclically monotone* if there are Borel-measurable functions

$\phi : X \to [-\infty, \infty[$ and $\psi : Y \to [-\infty, \infty[$ such that

$$\phi(x) + \psi(y) \leq c(x, y), \qquad \text{for } \textbf{every } x, y,$$

$$\phi(x) + \psi(y) = c(x, y), \qquad \text{for } \pi - \textbf{a.e. } x, y.$$

Obvious: strong c-cyclically monotone $\Rightarrow$ c-cyclically monotone
BUT: $\Leftarrow$ fails in general (π_α of Ambrosio-Pratelli).

Proposition (S.-Teichmann)

If c is lower semi-continuous and finitely valued, t.f.a.e. for $\pi \in \Pi(\mu, \nu)$ with $I_c(\pi) = \iint c d\pi < \infty$.

- π is optimal,
- π is strongly c-cyclically monotone,
- π is c-cyclically monotone.

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A crucial step in the proof of $(i) \Rightarrow (ii)$

It is known (Kellerer '84,...) that - under the above assumptions - there is no duality gap, i.e.

$$\lim_{n\to\infty} \left(\mathbb{E}_\mu[\varphi_n] + \mathbb{E}_\nu[\psi_n]\right) = \mathbb{E}_\pi(c),$$

for some sequence $(\varphi_n, \psi_n)_{n=1}^\infty$ of bounded Borel-measurable functions such that $\varphi_n(x) + \psi_n(y) \leq c(x,y)$.

How to pass to a limit?

Warning:

The limiting funtions φ, ψ (if we succeed in finding them) have no reason to be in $L^1(\mu)$ and $L^1(\nu)$ respectively. There are easy counterexamples, even for $c(x,y) = (x-y)^2/2$ and $X = Y = \mathbb{R}$.

Komlos type Lemma (Delbaen-S. 94):

Let $(f_n)_{n=1}^\infty$ be a sequence in $L^0_+(\Omega, \mathcal{F}, \mathbb{P})$. There exist convex combinations $g_n \in conv(f_n, f_{n+1}, \dots)$ such that $(g_n)_{n=1}^\infty$ converges almost surely.

Apply this lemma to the non-negative functions $(c - (\varphi_n + \psi_n))_{n=1}^\infty$.

Further cases where the answer to Villanis question is positive:

Pratelli:

When c is $[0, \infty]$-valued and **continuous**.

Beiglböck, Goldstern, Maresch, S.:

When c is Borel measurable and $\{c = \infty\} = F \cup N$, where F is closed in $X \times Y$ and F is a $\mu \times \nu$-null set.

The general picture (Beiglböck, Goldstern, Maresch, S.):
From now on c is (only) assumed to be Borel-measurable and π is a given element of $\Pi(\mu, \nu)$.

Example:

Let $X = Y = [0, 1], \mu = \nu$ the Lebesgue-measure and set

$$c(x, y) = \begin{cases} \infty & \text{for } x < y \\ 1 & \text{for } x = y \\ 0 & \text{for } x > y \end{cases}$$

for $(x, y) \in X \times Y$. The optimal (and in fact the only finite) transport plan π is concentrated on the diagonal and yields costs of one.

But, for every φ, ψ with $\varphi + \psi \leq c$ we have $\mathbb{E}_\mu[\varphi] + \mathbb{E}_\nu[\psi] \leq 0$.

There is a duality gap!

Remaining question:

What are general conditions on a Borel-measure $c : X \times Y \to [0, \infty]$, insuring that there is no duality gap? Formally

$$\inf_{\pi \in \Pi(\mu,\nu)} \mathbb{E}_\pi[c] = \sup_{\substack{\varphi,\psi \\ \varphi+\psi \le c}} (\mathbb{E}_\mu[\varphi] + \mathbb{E}_\nu[\psi])$$

The above example shows that, in general, there is a duality gap.

Lemma:

Let $\pi, \tilde{\pi} \in \Pi(\mu, \nu)$ and $\varphi : X \to [-\infty, \infty[$, $\psi : Y \to [-\infty, \infty[$ Borel-measurable such that $\mathbb{E}_\pi[\varphi + \psi] < \infty$ and $\mathbb{E}_{\tilde{\pi}}[\varphi + \psi] < \infty$. Then

$$\mathbb{E}_\pi[\varphi + \psi] = \mathbb{E}_{\tilde{\pi}}[\varphi + \psi].$$

In the case when $\varphi \in L^1(\mu), \psi \in L^1(\nu)$
we also have

$$\mathbb{E}_\pi[\varphi + \psi] = \mathbb{E}_\mu[\varphi] + \mathbb{E}_\nu[\psi].$$

For Borel-measurable $c : X \times Y \to [0, \infty]$ such that there is some $\pi_0 \in \Pi(\mu, \nu)$ with finite transport cost $\mathbb{E}_{\pi_0}[c]$, and φ, ψ as above with $\varphi + \psi \leq c$, may therefore well-define

$$J(\varphi, \psi) = \mathbb{E}_\pi[\varphi + \psi], \qquad \pi \in \Pi(\mu, \nu), \mathbb{E}_\pi[c] < \infty.$$

Theorem (Beiglböck-S.):

Assume that $c : X \times Y \to [0, \infty]$ is Borel-measurable and $\mu \times \nu$-a.s. finite, and suppose that there is $\pi_0 \in \Pi(\mu, \nu)$ with $\mathbb{E}_{\pi_0}[c] < \infty$.

Then there are Borel measurable functions

$$\hat{\phi} : X \to [-\infty, \infty[, \;\; \hat{\psi} : Y \to [-\infty, \infty[$$

such that

$$\hat{\varphi}(x) + \hat{\psi}(y) \leq c(x, y), \quad \text{for all } (x, y) \in X \times Y,$$

and

$$\inf_{\pi \in \Pi(\mu,\nu)} \mathbb{E}_\pi[c] = J(\hat{\varphi}, \hat{\psi}) = \sup_{\substack{\varphi, \psi \text{ Borel} \\ \varphi + \psi \leq c}} J(\varphi, \psi).$$

Proposition

Let X, Y be Polish spaces equipped with Borel probability measures μ, ν. Let $c : X \times Y \to [0, \infty]$ be Borel measurable, assume that π is a finite transport plan and set $\alpha = I_c[\pi] - I_c \geq 0$. Then there exists a function $f : X \times Y \to [0, \infty]$ such that $\int f \, d\pi = \alpha$ and, for all $(x_1, y_1), \ldots, (x_n, y_n) \in X \times Y$,

$$\sum_{i=1}^{n} c(x_{i+1}, y_i) + f(x_i, y_i) - c(x_i, y_i) \geq 0.$$

Proposition

Assume that X, Y are Polish spaces equipped with Borel probability measures μ, ν, that $\bar{c} : X \times Y \to (-\infty, \infty]$ is Borel measurable and $\mu \otimes \nu$-a.e. finite and that $\underline{c} : X \times Y \to [-\infty, \infty)$ is Borel measurable. If

$$\sum_{i=1}^{n} \bar{c}(x_{i+1}, y_i) - \underline{c}(x_i, y_i) \geq 0$$

for all $x_1, \ldots, x_n \in X$, $y_1, \ldots, y_n \in Y$, there exist Borel measurable functions $\phi : X \to [-\infty, \infty), \psi : Y \to [-\infty, \infty)$ and Borel sets $X' \subseteq X$, $Y' \subseteq Y$ of full measure such that

$$\underline{c}(x, y) \leq \phi(x) + \psi(y) \leq \bar{c}(x, y),$$

where the lower bound holds for $x \in X', y \in Y'$ and the upper bounded is valid for all $x \in X, y \in Y$.

A variant (Beiglböck-S.) of the Ambrosio-Pratelli Example:

In the above setting let

$$c(x,y) = \begin{cases} 1 \ \text{ if } x = y \\ 0 \ \text{ if } T_\alpha(x) = y, \quad x \in [0, \frac{1}{2}[\\ 2 \ \text{ if } T_\alpha(x) = y, \quad x \in [\frac{1}{2}, 1[\\ \infty \ \text{otherwise} \end{cases}$$

In this case π_0 and π_α are *both primal optimizers*.

Duality holds true, i.e.

$$1 = \mathbb{E}_{\pi_0}(c) = \sup\{\mathbb{E}_\mu[\varphi] + \mathbb{E}_\nu[\psi]\}$$

where the sup is taken over all Borel-measureable, integrable φ, ψ satisfying $\varphi + \psi \leq c$.
But we cannot pass to a limit: there is no dual optimizer $(\hat{\varphi}, \hat{\psi})$, i.e. Borel measurable functions $\hat{\varphi}, \hat{\psi}$ such that $\hat{\varphi} + \hat{\psi} \leq c$ and

$$\mathbb{E}_\pi[\hat{\varphi}, \hat{\psi}] = 1, \qquad \text{for finite transport plans } \pi.$$