A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options

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Abstract

We consider the problem of superhedging under volatility uncertainty for an investor allowed to dynamically trade the underlying asset, and statically trade European call options for all possible strikes with some given maturity. This problem is classically approached by means of the Skorohod Embedding Problem (SEP). Instead, we provide a dual formulation which converts the superhedging problem into a continuous martingale optimal transportation problem. We then show that this formulation allows to recover previously known results about Lookback options. In particular, our methodology induces a new presentation of the Azéma-Yor solution of the SEP.

Key words: Optimal control, volatility uncertainty, convex duality.

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1 Introduction

In a financial market allowing for the dynamic trading of some given underlying assets without restrictions, the fundamental theorem of asset pricing essentially states that the absence of arbitrage opportunities is equivalent to the existence of a probability measure under which the underlying asset process is a martingale. See Kreps [16], Harrison and Pliska [12], and Delbaen and Schachermayer [8]. Then, for the purpose of hedging, the

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only relevant information is the quadratic variation of the assets price process under such a martingale measure. Without any further assumption on the quadratic variation, the robust superhedging cost reduces to an obvious bound which can be realized by static trading on the underlying assets, see Cvitanić, Pham and Touzi [7] and Frey [11].

In this paper, we examine the problem of superhedging, under the condition of no-arbitrage, when the financial market also allows for the static trading of European call options. For simplicity, we consider the case where all available European call options have the same maturity T. However, we idealize the financial market assuming that such European call options are available for all possible strikes. Then any $T-$maturity vanilla derivative can be perfectly replicated by a portfolio of European calls, and therefore has an un-ambiguous no-arbitrage price in terms of the given prices of the underlying calls.

This problem is classically approached in the literature by means of the Skorohod Embedding Problem (SEP) which shows up naturally due to the Dubins-Schwartz time change result. The use of SEP techniques to solve the robust superhedging problem can be traced back to Hobson [13]. The survey paper by Hobson [14] is very informative and contains the relevant references on the subject.

In this paper, we develop an alternative approach which relates the robust superhedging problem to the literature on stochastic control, see Fleming and Soner [10], and more specifically, optimal transportation. Our context opens the door to an original new ramification in the theory of optimal transportation as it imposes naturally that the transportation be performed along a continuous martingale. Our first main result, reported in subsections 2.3 and 2.4, provides a formulation of the robust superhedging problem based on the Kantorovich duality in the spirit of Benamou and Brenier [4], see Villani [23].

We then specialize the discussion to Lookback derivatives. In this context, the robust superhedging problem is known to be induced by the Azéma-Yor solution of the SEP [1, 2]. A semi-static hedging strategy corresponding to this bound was produced by Hobson [13]. Our second main result, reported in Section 3, reproduces this bound by means of our dual formulation. In particular, this provides a new presentation of the Azéma-Yor solution of the SEP. We also recover in Section 4 the robust superhedging cost for the forward Lookback option which was also derived in Hobson [13].

2 Model-free bounds of derivatives securities

2.1 The probabilistic framework

Let $\Omega := \{\omega \in C([0,T],\mathbb{R}^d) : \omega_0 = 0\}$ be the canonical space equipped with the uniform norm $\|\omega\|_\infty := \sup_{0\le t\le T}|\omega_t|$, B the canonical process, $\mathbb{P}_0$ the Wiener measure, $\mathbb{F} := \{\mathcal{F}_t\}_{0\le t\le T}$ the filtration generated by B, and $\mathbb{F}^+ := \{\mathcal{F}_t^+, 0 \le t \le T\}$ the right limit of $\mathbb{F}$, where $\mathcal{F}_t^+ := \cap_{s>t}\mathcal{F}_s$.

Throughout the paper, X_0 is some given initial value in $\mathbb{R}_+^d$, and we denote

$$X_t := X_0 + B_t \quad \text{for} \quad t \in [0,T].$$

For all $\mathbb{F}-$progressively measurable process α with values in $\mathcal{S}_d^+$ (space of definite positive symmetric matrices) and satisfying $\int_0^T |\alpha_s| ds < \infty$, $\mathbb{P}_0-$a.s., we define the probability measures on $(\Omega, \mathcal{F})$:

$$\mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where} \quad X_t^\alpha := X_0 + \int_0^t \alpha_r^{1/2} dB_r, \ t \in [0,T], \ \mathbb{P}_0 - \text{a.s.}$$

Then X is a $\mathbb{P}^\alpha-$local martingale. Following [21], we denote by $\overline{\mathcal{P}}_S$ the collection of all such probability measures on $(\Omega, \mathcal{F})$. The quadratic variation process $\langle X \rangle = \langle B \rangle$ is universally defined under any $\mathbb{P} \in \overline{\mathcal{P}}_S$, and takes values in the set of all nondecreasing continuous functions from $\mathbb{R}_+$ to $\mathcal{S}_d^+$. Moreover, for all $\mathbb{P} \in \overline{\mathcal{P}}_S$, the quadratic variation $\langle B \rangle$ is absolutely continuous with respect to the Lebesgue measure. We denote its density by:

$$\hat{a}_t \ := \ \frac{d\langle B \rangle_t}{dt} \ \in \ \mathcal{S}_d^+, \ \mathbb{P} - \text{a.s. for all} \quad \mathbb{P} \in \overline{\mathcal{P}}_S.$$

Finally, we recall from [21] that

$$\begin{array}{c} \text{every } \mathbb{P} \in \overline{\mathcal{P}}_S \text{ satisfies the Blumenthal zero-one law} \\ \text{and the martingale representation property.} \end{array} \tag{2.1}$$

In this paper, we shall focus on the subset $\mathcal{P}_\infty^+$ of $\overline{\mathcal{P}}_S$ consisting of all measures $\mathbb{P}$ such that

$$X \quad \text{is a} \ \ \mathbb{P} - \text{uniformly integrable martingale with values in} \quad \mathbb{R}_+^d.$$

Definition 2.1 *We say that a property holds quasi-surely (q.s.) if it holds $\mathbb{P}-$a.s. for every $\mathbb{P} \in \mathcal{P}_\infty^+$.*

The restriction of the probability measures in $\mathcal{P}_\infty^+$ to those induced by non-negative processes X is motivated by our subsequent interpretation of the entries X^i as price processes of financial securities.

2.2 Model-free super-hedging problem

We first introduce the set portfolio strategies

$$\hat{\mathbb{H}}^2_{\text{loc}} := \cap_{\mathbb{P}\in\mathcal{P}_\infty^+} \mathbb{H}^2_{\text{loc}}(\mathbb{P}) \ \ \text{where} \ \ \mathbb{H}^2_{\text{loc}}(\mathbb{P}) := \left\{ H \in \mathbb{H}^0(\mathbb{P}) : \int_0^T |\hat{a}_t^{1/2} H_t|^2 dt < \infty, \ \mathbb{P} - \text{a.s.} \right\} \quad .$$

Under the self-financing condition, any $H \in \hat{\mathbb{H}}^2_{\rm loc}$ induces the portfolio value process

$$Y^H_t := Y_0 + \int_0^t H_s \cdot dB_s, \quad t \in [0,T]. \tag{2.2}$$

This stochastic integral is well-defined $\mathbb{P}-$a.s. for every $\mathbb{P} \in \mathcal{P}^+_\infty$, and should be rather denoted $Y^{H^{\mathbb{P}}}_t$ to emphasize its dependence on $\mathbb{P}$. In general, it may not be possible to aggregate the family $\{Y^{H^{\mathbb{P}}}, \mathbb{P} \in \mathcal{P}^+_\infty\}$ into a universal process Y^H such that $Y^H = Y^{H^{\mathbb{P}}}$, $dt \otimes d\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}^+_\infty$. See [21] for a wide discussion of this question.

Finally, in order to avoid possible arbitrage opportunities which may be induced by doubling strategies, we define the set of admissible strategies $\mathcal{H}$ by

$$\mathcal{H} \ := \ \left\{ H \in \hat{\mathbb{H}}^2_{\rm loc} : \text{ for all } \mathbb{P} \in \mathcal{P}^+_\infty, \ Y^H \geq M^{\mathbb{P}} \ \mathbb{P}-\text{a.s. for some } \mathbb{P}-\text{martingale } M^{\mathbb{P}} \right\},$$

where the martingale $M^{\mathbb{P}}$ may depend on $\mathbb{P}$ and the portfolio strategy H. Then, it follows from (2.2) that

$$Y^H \quad \text{is a } \mathbb{P}-\text{local martingale and } \mathbb{P}-\text{supermartingale, for all} \quad H \in \mathcal{H}, \ \mathbb{P} \in \mathcal{P}^+_\infty. \tag{2.3}$$

Let ξ be an $\mathcal{F}_T-$measurable random variable. The model-free superhedging problem is defined by:

$$U^0(\xi) \ := \ \inf \left\{ Y_0 : Y^H_1 \geq \xi, \text{ q.s. for some } H \in \mathcal{H} \right\}. \tag{2.4}$$

We call U^0 the model-free superhedging bound, and we recall its interpretation as the no-arbitrage upper bound on the market price of the derivative security ξ, for an investor who has access to continuous-time trading the underlying securities with price process X.

2.3 Dual formulation of the super-hedging bound

We denote by UC(Ω) the collection of all uniformly continuous maps from Ω_{X_0} to $\mathbb{R}$, where $X_0 \in (0,\infty)^d$ is a fixed initial value, and $\Omega_{X_0} := \{\omega \in C([0,T],\mathbb{R}^d_+) : \omega_0 = X_0\}$. The following result is a direct adaptation from Soner, Touzi and Zhang [20]. We observe that we may weaken the subsequent assumptions on the payoff function ξ, see Remarks 5.1 and 5.2 below. However, we do not pursue in this direction as the applications of this paper only involve smooth payoffs.

Theorem 2.1 *Let $\xi \in UC(\Omega)$ be such that $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}^+_\infty$. Then:*

$$U^0(\xi) \ = \ \sup_{\mathbb{P} \in \mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}[\xi].$$

Assume further that $U^0(\xi) \in \mathbb{R}$. Then there exists a unique process $H \in \hat{\mathbb{H}}^2_{\rm loc}$ and a unique family of nondecreasing predictable processes $\{K^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}^+_\infty\}$, with $K^{\mathbb{P}}_0 = 0$ for all $\mathbb{P} \in \mathcal{P}^+_\infty$, such that:

$$\xi \ = \ U^0(\xi) + \int_0^1 H_t \cdot dB_t - K^{\mathbb{P}}_1, \quad \mathbb{P} - a.s. \text{ for all } \mathbb{P} \in \mathcal{P}^+_\infty. \tag{2.5}$$

The proof is reported in Section 5.

Remark 2.1 A similar dual representation as in Theorem 2.1 was first obtained by Denis and Martini [9] in the bounded volatility case. Notice however that the family of non-dominated singular measures in [9] is not included in our set $\overline{\mathcal{P}}_S$, and does not allow for the existence of an optimal super-hedging strategy.

2.4 Calibration adjusted no-arbitrage bound

In this section we specialize the discussion to the one-dimensional case. This is consistent with the one-dimensional practical treatment of vanilla options on real financial markets.

We assume that, in addition to the continuous-time trading of the primitive securities, the investor can take static positions on $T-$maturity European call or put options with all possible strikes $K \geq 0$. Then, from Breeden and Litzenberger [5], the investor can identify that the $T-$marginal distribution of the underlying asset under the pricing measure is some probability measure $\mu \in M(\mathbb{R}_+)$, the set of all probability measures on $\mathbb{R}_+$.

For any scalar function $\lambda \in \mathbb{L}^1(\mu)$, the $T-$maturity European derivative defined by the payoff $\lambda(X_T)$ has an un-ambiguous no-arbitrage price

$$\mu(\lambda) \quad = \quad \int \lambda d\mu,$$

and can be perfectly replicated by buying and holding a portfolio of down-and-in Arrows of all strikes, with the density of Arrows $\lambda(K)$ at strike K. See Carr and Chou [6]. In particular, given the spot price $X_0 > 0$ of the underlying assets, the probability measure μ must satisfies:

$$\int x\mu(dx) = X_0.$$

We now define an improvement of the no-arbitrage upper-bound by accounting for the additional possibility of statically trading the European call options. Let

$$\Lambda^\mu := \left\{\lambda \in \mathbb{L}^1(\mu): \ \lambda(X_T)^- \in \cap_{\mathbb{P}\in\mathcal{P}^+_\infty}\mathbb{L}^1(\mathbb{P})\right\} \quad \text{and} \quad \Lambda^\mu_{\mathrm{UC}} := \Lambda^\mu \cap \mathrm{UC}(\mathbb{R}_+). \tag{2.6}$$

The improved no-arbitrage upper bound is defined by:

$$U^\mu(\xi) \quad := \quad \inf\left\{Y_0 : \overline{Y}^{H,\lambda}_1 \geq \xi, \text{ q.s. for some } H \in \mathcal{H} \text{ and } \lambda \in \Lambda^\mu_{\mathrm{UC}}\right\}, \tag{2.7}$$

where $\overline{Y}^{H,\lambda}$ denotes the portfolio value of a self-financing strategy with continuous trading H in the primitive securities, and static trading λ in the $T-$maturity European calls with all strikes:

$$\overline{Y}^{H,\lambda}_1 \quad := \quad Y^H_1 - \mu(\lambda) + \lambda(X_T), \tag{2.8}$$

indicating that the investor has the possibility of buying at time 0 any derivative security with payoff $\lambda(X_T)$ for the price $\mu(\lambda)$.

The next result is a direct application of Theorem 2.1.

Proposition 2.1 *Let $\xi \in UC(\Omega)$ be such that $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}^+_\infty$. Then, for all $\mu \in M(\mathbb{R}_+)$:*

$$U^\mu(\xi) \;=\; \inf_{\lambda\in\Lambda^\mu_{\rm UC}} \sup_{\mathbb{P}\in\mathcal{P}^+_\infty} \big\{\mu(\lambda) + \mathbb{E}^{\mathbb{P}}\big[\xi - \lambda(X_T)\big]\big\}.$$

Proof Observe that

$$U^\mu(\xi) \;=\; \inf_{\lambda\in\Lambda^\mu_{\rm UC}} U^0\big(\xi + \mu(\lambda) - \lambda(X_T)\big).$$

For every fixed λ, if $V(0) := \sup_{\mathbb{P}\in\mathcal{P}^+_\infty} \mathbb{E}^{\mathbb{P}}[\xi + \mu(\lambda) - \lambda(X_T)] < \infty$, then the previous proof of Theorem 2.1 applies and we get $U^0\big(\xi + \mu(\lambda) - \lambda(X_T)\big) = V(0)$. On the other hand, if $V(0) = \infty$, then notice from the proof of Theorem 2.1 that the inequality $U^0\big(\xi + \mu(\lambda) - \lambda(X_T)\big) \ge V(0)$ is still valid in this case, and therefore $U^0\big(\xi + \mu(\lambda) - \lambda(X_T)\big) = V(0)$. □

Remark 2.2 As a sanity check, let us consider the case $\xi = g(X_T)$, for some uniformly continuous function g with $\mu(|g|) < \infty$ and $\mathbb{E}^{\mathbb{P}}[g(X_T)] < \infty$ for all $\mathbb{P} \in \mathcal{P}^+_\infty$, and let us verify that $U^\mu(\xi) = \mu(g)$.

First, since $g \in \Lambda^\mu_{\rm UC}$, it follows from the dual formulation of Proposition 2.1 that $U^\mu(\xi) \le \mu(g)$. On the other hand, it is easily seen that

$$\sup_{\mathbb{P}\in\mathcal{P}^+_\infty} \mathbb{E}^{\mathbb{P}}[g(X_T)] \;=\; g^{\rm conc}(X_0),$$

where $g^{\rm conc}$ is the smallest concave majorant of g. Then, it follows from the dual formulation of Proposition 2.1 that $U^\mu(\xi) = \inf_{\lambda\in\Lambda^\mu_{\rm UC}} \mu(\lambda) + (g-\lambda)^{\rm conc}(X_0) \ge \inf_{\lambda\in\Lambda^\mu_{\rm UC}} \mu(\lambda) + \mu(g-\lambda) = \mu(g)$ as expected.

2.5 Connection with optimal transportation theory

As an alternative point of view, one may directly imbed in the no-arbitrage bounds the calibration constraint that the risk neutral marginal distribution of B_T is given by μ.

For convenience of comparison with the optimal transportation theory, the discussion of this subsection will be focused on the no-arbitrage lower bound. A natural formulation of the calibration adjusted no-arbitrage lower bound is:

$$\ell(\xi,\mu) := \inf\left\{\mathbb{E}^{\mathbb{P}}[\xi] :\; \mathbb{P} \in \mathcal{P}^+_\infty,\; X_0 \sim_{\mathbb{P}} \delta_{X_0},\; \text{and } X_T \sim_{\mathbb{P}} \mu\right\}. \tag{2.9}$$

where δ_{X_0} denotes the Dirac mass at the point X_0. We observe that a direct proof that $\ell(\xi,\mu)$ coincides with the corresponding sub-hedging cost is not obvious in the present context.

Under this form, the problem appears as minimizing the coupling criterion $E^{\mathbb{P}}[\xi]$ which involves the law of the process X under $\mathbb{P}$, over all those probability measures $\mathbb{P}\in\mathcal{P}_\infty^+$ such that the marginal distributions of X at times 0 and T are fixed. This is the general scope of optimal transportation problems as introduced by Monge and Kantorovitch, see e.g. Villani [23] and Mikami and Thieulen [17]. Motivated by the present financial application, Tan and Touzi [22] extended the Kantorovitch duality as described below. However, the above problem $\ell(\xi,\mu)$ does not satisfy the assumptions in [22] so that none of the results contained in this literature apply to our context.

The classical approach in optimal transportation consists in deriving a dual formulation for the problem (2.9) by means of the classical convex duality theory. Recall that $M(\mathbb{R}_+)$ denotes the collection of all probability measures on $\mathbb{R}_+$. Then, the Legendre dual with respect to μ is defined by

$$\ell^*(\xi,\lambda) := \sup_{\mu\in M(\mathbb{R}_+)} \{\lambda(\mu)-\ell(\xi,\mu)\} \quad \text{for all} \quad \lambda\in C_b^0(\mathbb{R}_+)$$

the set of all bounded continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$. Direct calculation shows that:

$$\begin{aligned}\ell^*(\xi,\lambda) &= \sup\left\{\mathbb{E}^{\mathbb{P}}[\lambda(X_T)-\xi]:\ \mu\in M(\mathbb{R}_+),\ \mathbb{P}\in\mathcal{P}_\infty^+,\ X_0\sim_{\mathbb{P}}\delta_{X_0},\ \text{and}\ X_T\sim_{\mathbb{P}}\mu\right\}\\ &= \sup\left\{\mathbb{E}^{\mathbb{P}}[\lambda(X_T)-\xi]:\ \mathbb{P}\in\mathcal{P}_\infty^+,\ X_0\sim_{\mathbb{P}}\delta_{X_0}\right\}.\end{aligned}$$

Observe that the latter problem is a standard (singular) diffusion control problem.

It is easily checked that ℓ is convex in μ. However, due to the absence of a uniform bound on the quadratic variation of X under $\mathbb{P}\in\mathcal{P}_\infty^+$, it is not obvious whether it is lower semicontinuous with respect to μ. If the latter property were true, then the equality $\ell^{**}=\ell$ provides

$$\ell(\xi,\mu) \;=\; \sup_{\lambda\in C_b^0}\{\mu(\lambda)-\ell^*(\xi,\lambda)\},$$

which is formally (up to the spaces choices) the lower bound analogue of the dual formulation of Proposition 2.1. A discrete-time analysis of this duality is contained in the parallel work to the present one by Beiglböck, Henry-labordère and Penkner [3].

3 Application to lookback derivatives

In this section, we consider derivative securities defined by the lookback payoff:

$$\xi = g\left(X_T^*\right), \quad \text{where} \quad X_T^* := \max_{t\le T} X_t, \tag{3.1}$$

and

$$g:\mathbb{R}_+\longrightarrow\mathbb{R}_+ \quad \text{is a nondecreasing } C^1 \text{ function} \tag{3.2}$$

Our main interest is to show that the optimal upper bound given by Proposition 2.1:

$$U^\mu(\xi) \;=\; \inf_{\lambda\in\Lambda^\mu_{\rm UC}}\big\{\mu(\lambda)+u^\lambda(0,X_0,X_0)\big\}$$

reproduces the already known bound corresponding to the Azema-Yor solution to the Skorohod embedding problem. Here, u^λ is the value function of the dynamic version of stochastic control problem

$$u^\lambda(t,x,m):=\sup_{\mathbb{P}\in\mathcal{P}^+_\infty}\mathbb{E}^{\mathbb{P}}\Big[g\big(M_T^{t,x,m}\big)-\lambda(X_T^{t,x})\Big],\quad t\le T,\quad (x,m)\in\mathbf{\Delta}, \tag{3.3}$$

where $\mathbf{\Delta}:=\big\{(x,m)\in\mathbb{R}^2_+:\ x\le m\big\}$, and

$$X_u^{t,x}:=x+(B_u-B_t),\quad M_u^{t,x,m}:=m\vee\max_{t\le r\le u}X_r^{t,x},\quad 0\le t\le u\le T.$$

When the time origin is zero, we shall simply write $X_u^x:=X_u^{0,x}$ and $M_u^{x,m}:=M_u^{0,x,m}$.

In the present context, the Markovian feature of the problem allows for an easy extension of the dual formulation of Proposition 2.1:

$$U^\mu(\xi) \;=\; \inf_{\lambda\in\Lambda^\mu}\big\{\mu(\lambda)+u^\lambda(0,X_0,X_0)\big\} \tag{3.4}$$

where Λ^μ is defined in (2.6), see Remark 5.1.

3.1 Formulation in terms of optimal stopping

We first convert the optimization problem u^λ into an infinite horizon optimal stopping problem.

Proposition 3.1 *For any $\lambda\in\Lambda^\mu$, the functions u^λ is independent of t and:*

$$u^\lambda(x,m) \;=\; \sup_{\tau\in\mathcal{T}^+_\infty}\mathbb{E}^{\mathbb{P}_0}\big[g(M_\tau^{x,m})-\lambda(X_\tau^x)\big]\quad \textit{for all}\quad (x,m)\in\mathbf{\Delta}, \tag{3.5}$$

where $\mathcal{T}^+_\infty$ is the collection of all stopping times τ such that the stopped process $\{X_{t\wedge\tau},t\ge 0\}$ is a non-negative $\mathbb{P}_0-$uniformly integrable martingale.

Proof From the regularity of g and λ, we may write the stochastic control problem (3.3) in its strong formulation

$$U^\lambda(t,x,m):=\sup_\sigma\mathbb{E}^{\mathbb{P}_0}\Big[g\big(M_T^{\sigma,t,x,m}\big)-\lambda(X_T^{\sigma,t,x})|(X_t^{\sigma,t,x},M_t^{\sigma,t,x,m})=(x,m)\Big],$$

where

$$X_s^{\sigma,t,x} = x + \int_t^s \sigma_r dB_r, \quad M_s^{\sigma,t,x,m} := m \vee \max_{t \le r \le s} X_r^{\sigma,t,x}, \quad 0 \le t \le s \le T,$$

and σ ranges in the set of all nonnegative processes $\mathbb{H}^2(\mathbb{P}_0)$ and such that the process $\{X_s^{\sigma,t,x}, t \le s \le T\}$ is a non-negative uniformly integrable martingale. Then, the required result is a direct consequence of the Dubins-Schwartz time change for martingales in the Brownian filtration. □

In view of the previous results, we are reduced to the problem:

$$U^\mu(\xi) \;\; := \;\; \inf_{\lambda \in \Lambda_0^\mu} \big\{\mu(\lambda) + u^\lambda(X_0, X_0)\big\}, \tag{3.6}$$

where:

$$u^\lambda(X_0, X_0) := \sup_{\tau \in \mathcal{T}_\infty^+} J(\lambda, \tau), \quad J(\lambda, \tau) := \mathbb{E}^{\mathbb{P}_0}\Big[g(X_\tau^*) - \lambda(X_\tau)\Big].$$

and the set Λ^μ of (2.6) translates in the present context to:

$$\Lambda_0^\mu \;\; = \;\; \big\{\lambda \in \mathbb{L}^1(\mu) : \; \lambda(X_\tau)^- \in \mathbb{L}^1(\mathbb{P}_0) \text{ for all } \tau \in \mathcal{T}_\infty^+\big\}. \tag{3.7}$$

3.2 The main result

The endpoints of the support of the distribution μ are denoted by:

$$\ell^\mu := \sup\big\{x : \mu\big([x,\infty)\big) = 1\big\} \quad \text{and} \quad r^\mu := \inf\big\{x : \mu\big((x,\infty)\big) = 0\big\}$$

The Azéma-Yor solution of the *Skorohod Embedding Problem* is defined by means of the so-called barycenter function:

$$b(x) := \frac{\int_x^\infty y\mu(dy)}{\mu\big([x,\infty)\big)} \, \mathbf{1}_{\{x<r^\mu\}} + x \, \mathbf{1}_{\{x \ge r^\mu\}} \quad x \ge 0. \tag{3.8}$$

Remark 3.1 D. Hobson [13] observed that the barycenter function can be alternatively defined as the right-continuous inverse to the following function β. Given the European calls prices $c(x) := \int (y-x)^+\mu(dy)$ and $X_0 = \int y\mu(dy)$, define the function

$$\beta(x) := \max\Big\{\operatorname{argmin}_{y<x} \tfrac{c(y)}{x-y}\Big\} \text{ for } \; x \in [X_0, r^\mu), \tag{3.9}$$

$$\beta(x) = 0 \;\text{ for }\; x \in [0, \ell^\mu) \;\text{ and }\; \beta(x) = x \;\text{ for }\; x \in [r^\mu, \infty). \tag{3.10}$$

On $[X_0, r^\mu)$, $\beta(x)$ is the largest minimizer of the function $y \longmapsto c(y)/(x-y)$ on $(0,x)$. Then, β is nondecreasing, right-continuous, and $\beta(x) < x$ for all $x \in [X_0, r^\mu]$. Notice that $\beta(X_0) = \ell^\mu := \sup\{x : \mu\big((0,x]\big) > 0\}$.

The following result is a combination of [13] and [19]. Our objective is to derive it directly from the dual formulation of Proposition 2.1. Let

$$\tau^* \quad := \quad \inf\big\{t > 0 : \ X_t^* \geq b(X_t)\big\}, \tag{3.11}$$

and

$$\lambda^*(x) := \int_0^x \int_0^y g'\big(b(\xi)\big)\frac{\mu(d\xi)}{\mu\big([\xi,\infty)\big)}dy; \quad 0 \leq x < r^\mu. \tag{3.12}$$

Notice that $\lambda^* \in [0,\infty]$ as the integral of a nonnegative function. To see that $\lambda^* < \infty$, we compute by the Fubini theorem that:

$$\lambda^*(x) \quad = \quad \int_0^x (x-\xi)g'\big(b(\xi)\big)\frac{\mu(d\xi)}{\mu\big([\xi,\infty)\big)},$$

so that the finiteness of λ^* is implied by that of its majorant $\int_0^x g'\big(b(\xi)\big)\mu(d\xi)/\mu\big([\xi,\infty)\big)$, for which we directly compute that it is equivalent to

$$\int_0^x g'(X_0)\frac{\mu(d\xi)}{\mu\big([\xi,\infty)\big)} = -g'(X_0)\ln\mu\big([x,\infty)\big) < \infty \quad \text{for all} \quad x \in [0, r^\mu).$$

Theorem 3.1 *Let ξ be given by* (3.1) *for some nondecreasing C^1 payoff function g, and $\mu \in M(\mathbb{R}_+)$ be such that $\mu(g \circ b) < \infty$. Then:*

$$U^\mu(\xi) \quad = \quad \mu(\lambda^*) + J(\lambda^*, \tau^*) \ = \ \mu(g \circ b).$$

The proof is reported in the subsequent subsection.

3.3 An upper bound for the optimal upper bound

In this section, we prove that:

$$U^\mu(\xi) \quad \leq \quad \mu(\lambda^*) + J(\lambda^*, \tau^*). \tag{3.13}$$

Our first step is to use the following construction due to Peskir [19] which provides a guess of the value function u^λ for functions λ in the subset:

$$\hat{\Lambda}_0^\mu \quad := \quad \{\lambda \in \Lambda_0^\mu : \lambda \text{ is convex}\}. \tag{3.14}$$

By classical tools from stochastic control theory, the value function $u^\lambda(x,m)$ is expected to solve the dynamic programming equation:

$$\min\big\{u^\lambda - g + \lambda, -u_{xx}^\lambda\big\} = 0 \ \text{ on } \ \mathbf{\Delta} \ \text{ and } \ u_m^\lambda(m,m) = 0 \ \text{ for } \ m \geq 0. \tag{3.15}$$

The first part of the above DPE is an ODE for which m appear only as a parameter involved in the domain on which the ODE must hold. Since we are restricting to convex λ, one can guess a solution of the form:

$$v^{\psi}(x,m) := g(m) - \lambda\big(x \wedge \psi(m)\big) - \lambda'\big(\psi(m)\big)\big(x - x \wedge \psi(m)\big), \tag{3.16}$$

i.e. $v^{\psi}(x,m) = g(m) - \lambda(x)$ for $x \in [0, \psi(m)]$ and is given by the tangent at the point $\psi(m)$ for $x \in [\psi(m), \infty)$. For later use, we observe that

$$\begin{aligned} v^{\psi}(x,m) &= g(m) - \lambda(\psi(m)) - \int_{\psi(m)}^{x} \frac{\partial}{\partial y}\{\lambda'(y)(x-y)\}dy \\ &= g(m) - \lambda(x) - \int_{\psi(m)}^{x} (x-y)\lambda''(dy) \quad \text{for} \quad x \geq \psi(m), \end{aligned} \tag{3.17}$$

where λ'' is the second derivative measure of the convex function λ.

We next choose the function ψ in order to satisfy the Neumann condition in (3.15). Assuming that λ is smooth, we obtain by direct calculation that the free boundary ψ must verify the ordinary differential equation (ODE):

$$\lambda''\big(\psi(m)\big)\psi'(m) = \frac{g'(m)}{m - \psi(m)} \quad \text{for all} \quad m \geq 0. \tag{3.18}$$

For technical reasons, we need to consider this ODE in the relaxed sense. This contrasts our analysis with that of Peskir [19] and Obloj [18]. Since λ is convex, its second derivative λ'' is well-defined as measure on $\mathbb{R}_+$. We then introduce the weak formulation of the ODE (3.18):

$$\int_{\psi(B)} \lambda''(dy) = \int_B \frac{g'(m)}{m - \psi(m)}\, dm \quad \text{for all} \quad B \in \mathcal{B}(\mathbb{R}_+), \tag{3.19}$$

and we introduce the collection of all relaxed solutions of (3.18):

$$\Psi^{\lambda} \;:=\; \big\{\psi \text{ right-continuous : } (3.19) \text{ holds and } \psi(x) < x \text{ for all } x \geq 0\big\}. \tag{3.20}$$

Remark 3.2 For later use, we observe that (3.19) implies the all functions $\psi \in \Psi^{\lambda}$ are non-decreasing, and by direct integration that

$$\text{the function} \quad x \longmapsto \lambda(x) - \textstyle\int_{X_0}^{x} \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)} d\xi dy \quad \text{is affine,}$$

where ψ^{-1} is the right-continuous inverse of ψ. This follows from direct differentiation of the above function in the sense of generalized derivatives.

A remarkable feature of the present problem is that there is no natural boundary condition for the ODE (3.18) or its relaxation (3.19). The following result extends the easy part of the elegant maximality principle proved in Peskir [19] by allowing for possibly nonsmooth functions λ. We emphasize the fact that our approach does not need the full strength of Peskir's maximality principle.

Lemma 3.1 *Let $\lambda \in \hat{\Lambda}_0^\mu$ and $\psi \in \Psi^\lambda$ be arbitrary. Then $u^\lambda \le v^\psi$.*

Proof We organize the proof in three steps.
1. We first prove that v^ψ is differentiable in m on the diagonal with

$$v_m^\psi(m,m) = 0 \quad \text{for all} \quad m \ge 0. \tag{3.21}$$

Indeed, since $\psi \in \Psi_\lambda$, it follows from Remark 3.2 that

$$\lambda(x) = c_0 + c_1 x + \int_{X_0}^{x} \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)} d\xi dy$$

for some scalar constants c_0, c_1. Plugging this expression into (3.16), we see that:

$$\begin{aligned} v^\psi(x,m) &= g(m) - \Big(c_0 + c_1\psi(m) + \int_{X_0}^{\psi(m)} \int_{X_0}^{\psi^{-1}(y)} \frac{g'(\xi)}{\xi - \psi(\xi)} d\xi dy\Big) \\ &\quad -\Big(c_1 + \int_{X_0}^{m} \frac{g'(\xi)}{\xi - \psi(\xi)} d\xi\Big)\big(x - \psi(m)\big) \\ &= g(m) - c_0 - c_1 x + \int_{X_0}^{m} \frac{g'(\xi)}{\xi - \psi(\xi)} (\psi(\xi) - x) d\xi, \end{aligned}$$

where the last equality follows from the Fubini Theorem together with the fact that g is nondecreasing and $\psi(\xi) < \xi$. Since g is differentiable, (3.21) follows by direct differentiation with respect to m.
2. For an arbitrary stopping time $\tau \in \mathcal{T}_\infty^+$, we introduce the stopping times $\tau_n := \tau \wedge \inf\{t > 0 : |X_t - x| > n\}$. Since v^ψ is concave in x, as a consequence of the convexity of λ, it follows from the Itô-Tanaka formula that:

$$\begin{aligned} v^\psi(x,m) &\ge v^\psi(X_{\tau_n}, M_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) dM_t \\ &\ge g(M_{\tau_n}) - \lambda(X_{\tau_n}) - \int_0^{\tau_n} v_x^\psi(X_t, M_t) dB_t - \int_0^{\tau_n} v_m^\psi(X_t, M_t) dM_t \end{aligned}$$

by the fact that $v^\psi \ge g - \lambda$. Notice that $(M_t - X_t) dM_t = 0$. Then by the Neumann condition (3.21), we have $v_m^\psi(X_t, M_t) dM_t = v_m^\psi(M_t, M_t) dM_t = 0$. Taking expectations in the last inequality, we see that:

$$v^\psi(x,m) \ge \mathbb{E}_{x,m}\left[g(M_{\tau_n}) - \lambda(X_{\tau_n})\right]. \tag{3.22}$$

3. We finally take the limit as $n \to \infty$ in the last inequality. First, recall that $(X_{t\wedge\tau})_{t\ge0}$ is a uniformly integrable martingale. Then, by the Jensen inequality, $\lambda(X_{\tau_n}) \ge \mathbb{E}[\lambda(X_\tau)|\mathcal{F}_{\tau_n}]$. Since $\lambda(X_\tau)^- \in \mathbb{L}^1(\mathbb{P}^0)$, this implies that $\mathbb{E}[\lambda(X_{\tau_n})] \ge \mathbb{E}[\lambda(X_\tau)]$ where we also used the tower property of conditional expectations. We then deduce from (3.22) that

$$v^\psi(x,m) \ge \lim_{n\to\infty} \mathbb{E}_{x,m}\left[g(M_{\tau_n}) - \lambda(X_\tau)\right] = \mathbb{E}_{x,m}\left[g(M_\tau) - \lambda(X_\tau)\right]$$

by the nondecrease of the process M and the function g together with the monotone convergence theorem. By the arbitrariness of $\tau \in \mathcal{T}_\infty^+$, the last inequality shows that $v^\psi \geq u^\lambda$. $\square$

Our next result involves the function:

$$\varphi(x,m) := \frac{c(x) - c_0(x)\mathbf{1}_{m<X_0}}{m-x} \quad \text{with} \quad c_0(x) := (X_0 - x)^+, \quad 0 \leq x < m, \tag{3.23}$$

and we recall that $c(x) := \int (\xi - x)^+ \mu(d\xi)$ is the (given) European call price with strike x.

Lemma 3.2 *For $\lambda \in \hat{\Lambda}_0^\mu$ and $\psi \in \Psi^\lambda$, we have:*

$$\mu(\lambda) + u^\lambda(X_0, X_0) \quad \leq \quad g(X_0) + \int \varphi\big(\psi(m), m\big) g'(m) dm.$$

Proof 1. Let $\alpha \in \mathbb{R}_+$ be an arbitrary point of differentiability of λ. Then

$$\lambda(x) \quad = \quad \lambda(\alpha) + \lambda'(\alpha)(x-\alpha) + \int_\alpha^x (x-y)\lambda''(dy).$$

Integrating with respect to $\mu - \delta_{X_0}$ and taking $\alpha < X_0$, this provides

$$\begin{aligned}
\mu(\lambda) - \lambda(X_0) \quad &= \quad \lambda'(\alpha)\Big(\int x\mu(dx) - X_0\Big) + \int \Big(\int_\alpha^x (x-y)\lambda''(dy)\Big)(\mu - \delta_{X_0})(dx) \\
&= \quad -\int_\alpha^x (X_0 - y)\lambda''(dy) + \int \mathbf{1}_{\{x \geq \alpha\}} \int_\alpha^x (x-y)^+ \lambda''(dy)\ \mu(dx) \\
&\qquad + \int \mathbf{1}_{\{x<\alpha\}} \int_x^\alpha (y-x)\lambda''(dy)\ \mu(dx).
\end{aligned}$$

Then sending α to 0, it follows from the convexity of λ together with the monotone convergence theorem that

$$\mu(\lambda) - \lambda(X_0) \quad = \quad \int (c - c_0)(y)\lambda''(dy).$$

2. By the inequality in Lemma 3.1 together with (3.17), we now compute that:

$$\begin{aligned}
\mu(\lambda) + u^\lambda(X_0, X_0) \quad &\leq \quad g(X_0) + \int \big(c(y) - c_0(y)(1_{\{y<X_0\}} - 1_{\{\psi(X_0)<y<X_0\}})\big)\lambda''(dy) \\
&= \quad g(X_0) + \int \big(c(y) - c_0(y) 1_{\{y<\psi(X_0)\}}\big)\lambda''(dy).
\end{aligned}$$

We next use the ODE (3.18) satisfied by ψ in the distribution sense. This provides:

$$\mu(\lambda) + u^\lambda(X_0, X_0) \quad \leq \quad g(X_0) + \int \frac{c(\psi(m)) - c_0(\psi(m))\mathbf{1}_{\{m<X_0\}}}{m - \psi(m)}\, g'(m) dm.$$

Here, we observe that the endpoints in the last integral can be taken to 0 and ∞ by the non-negativity of the integrand. $\square$

We now have all ingredients to express the upper bound (3.13) explicitly in terms of the barycenter function b of (3.8).

Lemma 3.3 *For a nondecreasing C^1 payoff function g, we have:*

$$\inf_{\lambda\in\Lambda_0^\mu}\big\{\mu(\lambda)+u^\lambda(X_0,X_0)\big\} \;\le\; \mu(g\circ b).$$

Proof Since $\hat\Lambda_0^\mu\subset\Lambda_0^\mu$, we compute from Lemma 3.2 that

$$\begin{aligned}\inf_{\lambda\in\Lambda_0^\mu}\big\{\mu(\lambda)+u^\lambda(X_0,X_0)\big\} &\le \inf_{\lambda\in\hat\Lambda_0^\mu}\big\{\mu(\lambda)+u^\lambda(X_0,X_0)\big\}\\ &\le g(X_0)+\inf_{\lambda\in\hat\Lambda_0^\mu}\inf_{\psi\in\Psi^\lambda}\int\varphi\big(\psi(m),m\big)g'(m)dm \qquad (3.24)\end{aligned}$$

In the next two steps, we prove that the last minimization problem on the right hand-side of (3.24) can be solved by pointwise minimization inside the integral. Then, in Step 3, we compute the induced upper bound.

1. For all $\lambda\in\hat\Lambda_0^\mu$ and $\psi\in\Psi^\lambda$:

$$\int\varphi\big(\psi(m),m\big)g'(m)dm \;\ge\; \int\inf_{\xi<m}\varphi\big(\xi,m\big)g'(m)dm$$

Observe that $c(x)\ge c_0(x)$ for all $x\ge 0$, and $\lim_{x\to 0}c(x)-c_0(x)=0$. Then

$$\inf_{\xi<m}\varphi(\xi,m)=\varphi(0,m)=0 \quad\text{for}\quad m<X_0. \qquad (3.25)$$

On the other hand, it follows from Remark 3.1 that:

$$\inf_{\xi<m}\varphi(\xi,m)=\inf_{\xi<m}\frac{c(\xi)}{m-\xi} \;=\; \frac{c\big(\beta(m)\big)}{m-\beta(m)} \quad\text{for}\quad m\ge X_0. \qquad (3.26)$$

By (3.25) and (3.26), we obtain the lower bound:

$$\int\varphi\big(\psi(m),m\big)g'(m)dm \;\ge\; \int\varphi\big(\beta(m),m\big)g'(m)dm.$$

2. We now observe that the function β, obtained by pointwise minimization in the previous step, solves the ODE (3.19). Therefore, in order to conclude the proof, it remains to verify that $\lambda^*\in\hat\Lambda_0^\mu$. The convexity of λ^* is obvious. Also, since $\lambda^*\ge 0$, we only need to prove that $\lambda^*\in\mathbb{L}^1(\mu)$. From the expressions of λ^* and b in (3.12) and (3.8), we compute that

$$\begin{aligned}\mu(\lambda^*) &= \int_0^\infty g'(b(\xi))(b(\xi)-\xi)\mu(d\xi)\\ &= \int_0^\infty g'(b(\xi))\int_\xi^\infty\mu(dx)db(\xi)\\ &= \int_0^\infty\big(g\circ b(x)-g\circ b(0)\big)\mu(dx) \;=\; \int_0^\infty\big(g(b(x))-g(X_0)\big)\mu(dx)\end{aligned}$$

by the Fubini theorem. hence the condition $\mu(g \circ b) < \infty$ is the precise translation of the integrability of λ^* with respect to μ.

3. From (3.24) and the previous two steps, we have:

$$
\begin{aligned}
\inf_{\lambda \in \Lambda_0^\mu} \left\{ \mu(\lambda) + u^\lambda(X_0, X_0) \right\} &\leq g(X_0) + \int_{X_0}^\infty \frac{c(\beta(x))}{x - \beta(x)} g'(x) dx \\
&= g(X_0) + \int_0^\infty \frac{c(y)}{b(y) - y} g'(b(y)) db(y) \\
&= g(X_0) + \int_0^\infty g'(b(y))(b(y) - y)\mu(dy) \;=\; \mu(g \circ b),
\end{aligned}
$$

where we used the fact that $c(y) = \int_y^\infty (\xi - y)\mu(d\xi)$ together with the calculation in Step 2.

□

3.4 Proof of Theorem 3.1

To complete the proof of the theorem, it remains to prove that

$$
\inf_{\lambda \in \Lambda_0^\mu} \left\{ \mu(\lambda) + u^\lambda(X_0, X_0) \right\} \;\geq\; \mu(g \circ b).
$$

To see this, we use the fact that the stopping time τ^* defined in (3.11) is a solution of the Skorohod embedding problem, i.e. $X_{\tau^*} \sim \mu$ and $(X_{t \wedge \tau^*})_{t \geq 0}$ is a uniformly integrable martingale, see Azéma and Yor [1, 2]. Then, for all $\lambda \in \Lambda_0^\mu$, it follows from the definition of u^λ that $u^\lambda(X_0, X_0) \geq J(\lambda, \tau^*)$, and therefore:

$$
\mu(\lambda) + u^\lambda(X_0, X_0) \;\geq\; \mu(\lambda) + \mathbb{E}_{X_0, X_0}\big[g(X^*_{\tau^*}) - \lambda(X_{\tau^*})\big] \;=\; \mathbb{E}_{X_0, X_0}\big[g(X^*_{\tau^*})\big].
$$

By the definition of τ^*, we have $X^*_{\tau^*} = b(X_{\tau^*})$. since $X_{\tau^*} \sim \mu$, this provides:

$$
\mu(\lambda) + u^\lambda(X_0, X_0) \;\geq\; \mathbb{E}_{X_0, X_0}\big[g \circ b(X_{\tau^*})\big] \;=\; \mu(g \circ b).
$$

4 Forward start lookback options

In this section, we provide a second application to the case where the derivative security is defined by the payoff

$$
\xi = g\big(B^*_{t_1, t_2}\big) \quad \text{where} \quad B^*_{t_1, t_2} := \max_{t_1 \leq t \leq t_2} B_t,
$$

and g satisfies the same conditions as in the previous section. We assume that the prices of call options $c_1(k)$ and $c_2(k)$ for the maturities t_1 and t_2 are given for all strikes:

$$
c_1(k) = \int (x - k)^+ \mu_1(dx) \quad \text{and} \quad c_2(k) = \int (x - k)^+ \mu_2(dx), \quad k \geq 0.
$$

We also assume that $\mu_1 \preceq \mu_2$ are in convex order:

$$c_1(0) = c_2(0) \quad \text{and} \quad c_1(k) \le c_2(k) \quad \text{for all} \quad k \ge 0.$$

The model-free superhedging cost is defined as the minimal initial capital which allows to superhedge the payoff ξ, quasi-surely, by means of some dynamic trading strategy in the underlying stock, and a static strategy in the calls $(c_1(k))_{k\ge 0}$ and $(c_2(k))_{k\ge 0}$.

This problem was solved in Hobson [13] in the case $g(x) = x$. Our objective here is to recover his results by means of our simple stochastic control approach.

A direct adaptation of Proposition 2.1 provides the dual formulation of this problem as:

$$U^{\mu_1,\mu_2}(\xi) \;\; = \;\; \sup_{(\lambda_1,\lambda_2)\in\Lambda^{\mu_1}\times\Lambda^{\mu_2}} \mu_1(\lambda_1) + \mu_2(\lambda_2) + u^{\lambda_1,\lambda_2}(X_0, X_0),$$

where

$$u^{\lambda_1,\lambda_2}(x,m) \;\; := \;\; \sup_{\mathbb{P}\in\mathcal{P}^+_\infty} \mathbb{E}^{\mathbb{P}}_{x,m}\big[g(B^*_{t_1,t_2}) - \lambda_1(B_{t_1}) - \lambda_2(B_{t_2})\big],$$

We next observe that the dynamic value function corresponding to the stochastic control problem u^{λ_1,λ_2} reduces to our previously studied problem u^{λ_2} at time t_1. Then, it follows from the dynamic programming principle that:

$$U^{\mu_1,\mu_2}(\xi) \;\; = \;\; \inf_{(\lambda_1,\lambda_2)\in\Lambda^{\mu_1}\times\Lambda^{\mu_2}} \mu_1(\lambda_1) + \mu_2(\lambda_2) + \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^{\mathbb{P}}\big[u^{\lambda_2}(B_{t_1}, B_{t_1}) - \lambda_1(B_{t_1})\big].$$

Since the expression to be maximized only involves the distribution of B_{t_1}, it follows from Remark 2.2 together with the Dubins-Schwartz time change formula that:

$$U^{\mu_1,\mu_2}(\xi) \;\; = \;\; \inf_{\lambda_2\in\Lambda_0^{\mu_2}} \mu_2(\lambda_2) + \int u^{\lambda_2}(x,x)\mu_1(dx),$$

We next obtain an upper bound by restricting attention to the subset $\hat{\Lambda}_0^{\mu_2}$ of convex multipliers of $\Lambda_0^{\mu_2}$. For such multipliers, we use the inequality $u^{\lambda_2} \le v^{\psi_2}$ for all $\psi_2 \in \Psi^{\lambda_2}$ as derived in Lemma 3.1. This provides:

$$\begin{aligned}
U^{\mu_1,\mu_2}(\xi) \;\; &\le \;\; \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \mu_2(\lambda_2) + \int v^{\psi_2}(x,x)\mu_1(dx) \\
&= \;\; \mu_1(g) + \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2\in\Psi^\lambda} \mu_2(\lambda_2) - \mu_1(\lambda_2) + \int_0^\infty \int_{\psi_2(x)}^x (x-y)\lambda_2''(dy)\mu_1(dx) \\
&= \;\; \mu_1(g) + \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2\in\Psi^\lambda} \int_0^\infty \Big(c_2(y) - c_1(y) + \int (x-y)\mathbf{1}_{\{\psi_2(x)<y<x\}}\mu_1(dx)\Big)\lambda_2''(dy) \\
&= \;\; \mu_1(g) + \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2\in\Psi^\lambda} \int_0^\infty \Big(c_2(y) - \int (x-y)\mathbf{1}_{\{y\le\psi_2(x)\}}\mu_1(dx)\Big)\lambda_2''(dy) \\
&= \;\; \mu_1(g) + \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2\in\Psi^\lambda} \int_0^\infty \Big(c_2(\psi_2(m)) - \int (x-\psi_2(m))\mathbf{1}_{\{m\le x\}}\mu_1(dx)\Big)\frac{g'(m)dm}{m-\psi_2(m)} \\
&= \;\; \mu_1(g) + \inf_{\lambda_2\in\hat{\Lambda}_0^{\mu_2}} \inf_{\psi_2\in\Psi^\lambda} \int\Big(\frac{c_2\big(\psi_2(m)\big) - c_1(m)}{m-\psi_2(m)} - \mu_1\big([m,\infty)\big)\Big)g'(m)dm
\end{aligned}$$

where the last equalities follow from similar manipulations as in Lemma 3.2, and in particular make use of the ODE (3.19). Since $g' \geq 0$, we may prove, as in the case of Lookback options, that the above minimization problem reduces to the pointwise minimization of the integrand, so that the optimal obstacle is given by:

$$\psi_2^*(x) = \max\left\{\operatorname{Arg}\min_{\xi<x} h(\xi)\right\} \quad \text{where} \quad h(\xi) := \frac{c_2(\xi) - c_1(m)}{m - \xi}, \quad \xi < m,$$

Notice that h has left and right derivative at every $\xi < m$, with

$$h'(\xi) \;=\; \frac{c_2(\xi) + (x-\xi)c_2'(\xi) - c_1(x)}{(x-\xi)^2}, \quad \text{a.e.}$$

where the numerator is a non-decreasing function of ξ, takes the positive value $c_2(x) - c_1(x)$ at $\xi = x$, and takes the negative value $X_0 - x - c_1(x)$ at $\xi = 0$. Then $\psi_2^*(x)$ is the largest root of the equation:

$$c_2\big(\psi_2^*(x)\big) + \big(x - \psi_2^*(x)\big)c_2'\big(\psi_2^*(x)\big) \;=\; c_1(x), \quad \text{a.e.} \tag{4.1}$$

so that h is nonincreasing to the left of $\psi_2^*(m)$ and nondecreasing to its right.

At this point, we recognize exactly the solution derived by Hobson [13]. In particular, ψ_2^* induces a solution τ_2^* to the Skorohod embedding problem, and we may use the expression of u^{λ_2} as the value function of an optimal stopping problem. Then, we may conclude the proof that the upper bound derived above is the optimal upper bound by arguing as in Section 3.4 that:

$$u^{\lambda_2}(x,x) \;\geq\; \mathbb{E}_{x,x}\big[g(X^*_{\tau_2^*}) - \lambda_2(X_{\tau_2^*})\big].$$

We get that the upper bound is given by:

$$\begin{aligned} U^{\mu_1,\mu_2}(\xi) &= \mu_1(g) - \int g'(m)\mu_1\big([m,\infty)\big)dm + \int \Big(\frac{c_2\big(\psi_2^*(m)\big) - c_1(m)}{m - \psi_2^*(m)}\Big)g'(x)dx \\ &= \mu_1(g) - \int \Big(c_2'\big(\psi_2^*(m)\big) - c_1'(m)\Big)g'(m)dm \\ &= g(0) - \int c_2'\big(\psi_2^*(m)\big)g'(m)dm \end{aligned} \tag{4.2}$$

by (4.1).

5 Proof of the duality result

Let $\xi : \Omega \longrightarrow \mathbb{R}$ be a measurable map with $\xi^+ \in \mathbb{L}^1(\mathbb{P})$ for all $\mathbb{P} \in \mathcal{P}_\infty^+$. Then $\mathbb{E}^{\mathbb{P}}[\xi] \in \mathbb{R} \cup \{-\infty\}$ is well defined. Let $X_0 \in \mathbb{R}$ be such that

$$X_1^H \geq \xi \quad \text{for some} \quad H \in \mathcal{H}. \tag{5.1}$$

By definition of the admissibility set $\mathcal{H}$, it follows that the process X^H is a $\mathbb{P}-$local martingale and a $\mathbb{P}-$supermartingale for any $\mathbb{P} \in \mathcal{P}_\infty$. Then, it follows from (5.1) that $X_0 \geq \mathbb{E}^{\mathbb{P}}[\xi]$ for all $\mathbb{P} \in \mathcal{P}_\infty$. From the arbitrariness of X_0 and $\mathbb{P}$, this shows that

$$U^0(\xi) \;\; \geq \;\; \sup_{\mathbb{P} \in \mathcal{P}_\infty^+} \mathbb{E}^{\mathbb{P}}[\xi]. \tag{5.2}$$

In the subsequent subsections, we prove that the converse inequality holds under the additional requirement that $\xi \in \mathrm{UC}(\Omega)$. In view of (5.2), it only remains to consider the case

$$\sup_{\mathbb{P} \in \mathcal{P}_\infty^+} \mathbb{E}^{\mathbb{P}}[\xi] \;\; < \;\; \infty. \tag{5.3}$$

Following [20], this result is obtained by introducing a *dynamic version* of the problem which is then proved to have a decomposition leading to the required result. Due to the fact that family of probability measures $\mathcal{P}_\infty^+$ is non-dominated, we need to define conditional distributions on all of the probability space without excepting any zero measure set.

5.1 Regular conditional probability distribution

Let $\mathbb{P}$ be an arbitrary probability measure on Ω, and τ be an $\mathbb{F}-$stopping time. The regular conditional probability distribution (r.c.p.d.) $\mathbb{P}_\tau^\omega$ is defined by

- For all $\omega \in \Omega$, $\mathbb{P}_\tau^\omega$ is a probability measure on $\mathcal{F}_1$,
- For all $E \in \mathcal{F}_1$, the mapping $\omega \longmapsto \mathbb{P}_\tau^\omega(E)$ is $\mathcal{F}_\tau-$measurable,
- For every bounded $\mathcal{F}_1$-measurable random variable ξ, we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi]$, $\mathbb{P}-$a.s.
- For all $\omega \in \Omega$, $\mathbb{P}_\tau^\omega\big[\omega' \in \Omega : \omega'(s) = \omega(s), 0 \leq s \leq \tau(\omega)\big] = 1$.

The existence of the r.c.p.d. is justified in Stroock and Varadhan [24]. For a better understanding of this notion, we introduce the shifted canonical space

$$\Omega^t := \{\omega \in C([t,1], \mathbb{R}^d) : \omega(t) = 0\} \quad \text{for all} \quad t \in [0,1],$$

we denote by B^t the shifted canonical process on Ω^t, $\mathbb{P}_0^t$ the shifted Wiener measure, and $\mathbb{F}^t$ the shifted filtration generated by B^t. For $0 \leq s \leq t \leq 1$ and $\omega \in \Omega^s$:

- the shifted path $\omega^t \in \Omega^t$ is defined by:

$$\omega_r^t := \omega_r - \omega_t \quad \text{for all} \quad r \in [t,1],$$

- the concatenation path $\omega \otimes_t \tilde{\omega} \in \Omega^s$, for some $\tilde{\omega} \in \Omega^t$, is defined by:

$$(\omega \otimes_t \tilde{\omega})(r) := \omega_r \mathbf{1}_{[s,t)}(r) + (\omega_t + \tilde{\omega}_r)\mathbf{1}_{[t,1]}(r) \quad \text{for all} \quad r \in [s,1].$$

- the shifted $\mathcal{F}_1^t-$ measurable r.v. $\xi^{t,\omega}$ of some $\mathcal{F}_1^s-$measurable r.v. ξ on Ω^s is defined by:

$$\xi^{t,\omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}) \quad \text{for all} \quad \tilde{\omega} \in \Omega^t.$$

Similarly, for an $\mathbb{F}^s-$progressively measurable process X on $[s,1]$, the shifted process $\{X^{t,\omega}_r, r \in [t,1]\}$ is $\mathbb{F}^t-$progressively measurable.

For notational simplicity, we set:

$$\omega \otimes_\tau \tilde{\omega} := \omega \otimes_{\tau(\omega)} \tilde{\omega}, \quad \xi^{\tau,\omega} := \xi^{\tau(\omega),\omega}, \quad X^{\tau,\omega} := X^{\tau(\omega),\omega}.$$

The r.c.p.d. $\mathbb{P}^\omega_\tau$ induces a probability measure $\mathbb{P}^{\tau,\omega}$ on $\mathcal{F}_1^{\tau(\omega)}$ such that the $\mathbb{P}^{\tau,\omega}-$distribution of $B^{\tau(\omega)}$ is equal to the $\mathbb{P}^\omega_\tau-$distribution of $\{B_t - B_{\tau(\omega)}, t \in [\tau(\omega), 1]\}$. Then, the r.c.p.d. can be understood by the identity:

$$\mathbb{E}^{\mathbb{P}^\omega_\tau}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau,\omega}}[\xi^{\tau,\omega}] \quad \text{for all} \quad \mathcal{F}_1 - \text{measurabler.v. } \xi.$$

We shall also call $\mathbb{P}^{\tau,\omega}$ the r.c.p.d. of $\mathbb{P}$.

For $0 \le t \le 1$, we follow the same construction as in Section 2.1 to define the martingale measures $\mathbb{P}^{t,\alpha}$ for each $\mathbb{F}^t-$progressively measurable $\mathcal{S}_d^{>0}-$valued process α such that $\int_t^1 |\alpha_r| dr < \infty$, $\mathbb{P}^t_0-$a.s. The collection of all such measures is denoted $\overline{\mathcal{P}}^t_S$. The subset $\mathcal{P}^t_\infty$ and the density process $\hat{a}^t$ of the quadratic variation process $\langle B^t \rangle$ are also defined similarly.

5.2 The duality result for uniformly continuous payoffs

Since $\xi \in \mathrm{UC}(\Omega)$, there exists a modulus of continuity function ρ such that for all $t \in [0,1]$ and $\omega, \omega' \in \Omega$, $\tilde{\omega} \in \Omega^t$,

$$\left| \xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega}) \right| \quad \le \quad \rho(\|\omega - \omega'\|_t),$$

where $\|\omega\|_t := \sup_{0 \le s \le t} |\omega_s|$, $0 \le t \le 1$. The main object in the present proof is the following dynamic value process

$$V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}^t_\infty} \mathbb{E}^{\mathbb{P}^\omega_t}[\xi] \quad \text{for all} \quad (t,\omega) \in [0,1] \times \Omega. \tag{5.4}$$

It follows from the uniform integrability property of ξ that

$$\{V_t, t \in [0,1]\} \quad \text{is a right-continuous } \mathbb{F}-\text{adapted process} \tag{5.5}$$

Moreover, by following exactly the proof of Proposition 4.7 in [20], we see that $\{V_t, t \in [0,1]\}$ satisfies the dynamic programming principle:

$$V_t(\omega) = \sup_{\mathbb{P} \in \mathcal{P}^+_\infty} \mathbb{E}^{\mathbb{P}^\omega_t}[V_s] \quad \text{for all} \quad 0 \le t \le s \le 1 \quad \text{and} \quad \omega \in \Omega. \tag{5.6}$$

Then, for all $\mathbb{P} \in \mathcal{P}^+_\infty$, the process $\{V_t, t \in [0,1]\}$ is a $\mathbb{P}-$supermartingale. By the Doob-Meyer decomposition, there exists a pair of processes $(H^\mathbb{P}, K^\mathbb{P})$, with $H^\mathbb{P} \in \mathbb{H}^2_{\mathrm{loc}}(\mathbb{P})$ and $K^\mathbb{P}$ $\mathbb{P}-$integrable nondecreasing, such that

$$V_t \quad = \quad V_0 + \int_0^t H^\mathbb{P}_s dB_s - K^\mathbb{P}_t, \quad t \in [0,1], \ \mathbb{P} - \text{a.s.}$$

Since V is a right-continuous semimartingale under each $\mathbb{P} \in \mathcal{P}_\infty^+$, it follows from Karandikar [15] that the family of processes $\{H^\mathbb{P}, \mathbb{P} \in \mathcal{P}_\infty^+\}$ (defined $\mathbb{P}-$a.s.) can be aggregated into a process $\hat{H}$ defined on $[0,1]\times\Omega$ by $d\langle V, B\rangle_t = \hat{H}_t d\langle B\rangle_t$, in the sense that $\hat{H} = H^\mathbb{P}$, $dt\times d\mathbb{P}-$a.s. for all $\mathbb{P} \in \mathcal{P}_\infty^+$. Thus we have

$$V_t \;=\; V_0 + \int_0^t \hat{H}_s dB_s - K_t^\mathbb{P}, \quad t \in [0,1], \; \mathbb{P}-\text{a.s. for all } \mathbb{P} \in \mathcal{P}_\infty^+.$$

With $X_0 := V_0$, we see that

- the process $X^{\hat{H}} := X_0 + \int_0^\cdot \hat{H}_s dB_s$ is bounded from below by V which is in turn bounded from below by $M_t^\mathbb{P} := \mathbb{E}_t^\mathbb{P}[\xi]$, $t \in [0,1]$; under (5.3) and the lower bound on ξ, the latter is a $\mathbb{P}-$martingale,
- and $X_1^{\hat{H}} = V_1 + K_1^\mathbb{P} = \xi + K_1^\mathbb{P} \geq \xi$, $\mathbb{P}-$a.s. for every $\mathbb{P} \in \mathcal{P}_\infty^+$.

Then $V_0 \geq U^0(\xi)$ by the definition of U^0.

Notice that, as a consequence of the supermartingale property of $X^{\hat{H}}$ under every $\mathbb{P} \in \mathcal{P}_\infty^+$, we have:

$$V_0 + \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^\mathbb{P}\big[-K_1^\mathbb{P}\big] \;\geq\; \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^\mathbb{P}\big[X_1^{\hat{H}} - K_1^\mathbb{P}\big] \;=\; \sup_{\mathbb{P}\in\mathcal{P}_\infty} \mathbb{E}^\mathbb{P}[\xi] \;=\; V_0.$$

Since $K_0^\mathbb{P} = 0$ and $K^\mathbb{P}$ is nondecreasing, this implies that

$$X^{\hat{H}} \quad \text{is a } \mathbb{P}-\text{martingale for all} \quad \mathbb{P} \in \mathcal{P}_\infty^+,$$

and the nondecreasing process $K^\mathbb{P}$ satisfies the minimality condition

$$\inf_{\mathbb{P}\in\mathcal{P}_\infty^+} \mathbb{E}^\mathbb{P}\big[K_1^\mathbb{P}\big] \;=\; 0.$$

Remark 5.1 A possible extension of Theorem 2.1 can be obtained for a larger class of payoff functions ξ. Indeed, notice that the uniform continuity assumption on ξ is only used to obtain properties (5.5) and (5.6) of the dynamic value process V defined in (5.4). In the context of the application of Section 3, a verification argument allows to express V_t explicitly as a smooth function of B_t and B_t^*. Then (5.5) and (5.6) hold true even if ξ is not uniformly continuous.

Remark 5.2 Another extension of Theorem 2.1 can be obtained by considering the closure of UC(Ω) with respect to a convenient norm. Since the applications of this paper only involve smooth payoffs, we do not pursue in this direction, and we refer the interested reader to [20].

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