

TOPICS IN EQUILIBRIUM TRANSPORT

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Based on joint works with A. Dupuy, Y.-W. Hsieh, S. Jaffe, S. Kominers,
and S. Weber

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- ▶ This talk is based on the following three working papers:
 - ▶ [GKW] AG, Kominers, S. and Weber, S.. Costly Concessions: An Empirical Framework for Matching with Imperfectly Transferable Utility.
 - ▶ [CGJK] Dupuy, A., AG, Jaffe, S., and Kominers, S. On the incidence of taxation in matching markets.
 - ▶ [GH] Galichon, A., Hsieh, Y.-W. A theory of decentralized matching markets without transfers, with an application to surge pricing.

Agenda:

1. Economic motivation
2. Equilibrium transportation
3. Galois connections and distance functions

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Section 1

ECONOMIC MOTIVATION

- ▶ Consider a model of the labor market. Assume that a population of *workers* is characterized by their type $x \in \mathcal{X}$, where $\mathcal{X} = \mathbb{R}^d$ for simplicity. There is a distribution P over the workers, which is assumed to sum to one.
- ▶ A population of *firms* is characterized by their types $y \in \mathcal{Y}$ (say $\mathcal{Y} = \mathbb{R}^d$), and their distribution Q . It is assumed that there is the same total mass of workers and firms, so Q sums to one.
- ▶ Each worker must work for one firm; each firm must hire one worker. Let $\pi(x, y)$ be the probability of observing a matched (x, y) pair. π should have marginal P and Q , which is denoted

$$\pi \in \mathcal{M}(P, Q).$$

- ▶ In the simplest case, the utility of a worker x working for a firm y at wage $w(x, y)$ will be

$$\alpha(x, y) + w(x, y)$$

while the corresponding profit of firm y is

$$\gamma(x, y) - w(x, y).$$

- ▶ In this case, the total surplus generated by a pair (x, y) is

$$\alpha(x, y) + w + \gamma(x, y) - w = \alpha(x, y) + \gamma(x, y) =: \Phi(x, y)$$

which does not depend on w (no transfer frictions). A central planner may thus like to choose assignment $\pi \in \mathcal{M}(P, Q)$ so to

$$\max_{\pi \in \mathcal{M}(P, Q)} \int \Phi(x, y) d\pi(x, y).$$

But as it turns out, this is also **the equilibrium solution**.

- ▶ The equilibrium assignment is determined by an important quantity: the **wages**. Let $w(x, y)$ be the wage of employee x working for firm of type y .
- ▶ Let the indirect surpluses of worker x and firm y be respectively

$$u(x) = \max_y \{ \alpha(x, y) + w(x, y) \}$$

$$v(y) = \max_x \{ \gamma(x, y) - w(x, y) \}$$

so that (π, w) is an equilibrium when

$$u(x) \geq \alpha(x, y) + w(x, y) \text{ with equality if } (x, y) \in \text{Supp}(\pi)$$

$$v(y) \geq \gamma(x, y) - w(x, y) \text{ with equality if } (x, y) \in \text{Supp}(\pi)$$

- ▶ By summation,

$$u(x) + v(y) \geq \Phi(x, y) \text{ with equality if } (x, y) \in \text{Supp}(\pi).$$

- ▶ One can show that the equilibrium outcome (π, u, v) is such that π is solution to the primal Monge-Kantorovich Optimal Transportation problem

$$\max_{\pi \in \mathcal{M}(P, Q)} \int \Phi(x, y) d\pi(x, y)$$

and (u, v) is solution to the dual OT problem

$$\begin{aligned} \min_{u, v} \int u(x) dP(x) + \int v(y) dQ(y) \\ \text{s.t. } u(x) + v(y) \geq \Phi(x, y) \end{aligned}$$

- ▶ Feasibility+Complementary slackness yield the desired equilibrium conditions

$$\begin{aligned} \pi &\in \mathcal{M}(P, Q) \\ u(x) + v(y) &\geq \Phi(x, y) \\ (x, y) \in \text{Supp}(\pi) &\implies u(x) + v(y) = \Phi(x, y) \end{aligned}$$

Here, **optimum=equilibrium**. “Second welfare theorem”, “invisible hand”, etc.

- ▶ Is equilibrium always the solution to an optimization problem?
- ▶ **It is not.** This is why this talk is about “Equilibrium Transportation,” which contains, but is strictly more general than “Optimal Transportation”.

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- ▶ Consider the same setting as above, but introduce (nonlinear) taxes.
- ▶ Instead of assuming that workers' and firm's payoffs are linear in wages, assume

$$u(x) = \max_y \{ \alpha_{xy} + N(w(x, y)) \}$$

$$v(y) = \max_x \{ \gamma_{xy} - w(x, y) \}$$

where $N(w)$ is indecreasing and continuous, interpreted as the net wage if w if the gross wage.

- ▶ Of course, OT is recovered when $N(w) = w$ (no tax).

- ▶ Let \mathcal{F}_{xy} be the set of feasible utilities that x and y can achieve through some wage w . One has

$$\mathcal{F}_{xy} = \{(u, v) : \exists w \in \mathbb{R}, u \leq \alpha_{xy} + N(w), v \leq \gamma_{xy} - w\},$$

which rewrites

$$\mathcal{F}_{xy} = \{(u, v) : u - \alpha_{xy} \leq N(\gamma_{xy} - v)\}.$$

- ▶ The interior of this set, denoted \mathcal{F}_{xy}^0 , is the set such that this inequality holds true.
- ▶ In the case of OT,

$$\mathcal{F}_{xy} = \{(u, v) : u + v \leq \alpha_{xy} + \gamma_{xy}\}.$$

Section 2

EQUILIBRIUM TRANSPORTATION

- ▶ We have therefore that (π, u, v) is an equilibrium outcome when

$$\begin{cases} (PF) : \pi \in \mathcal{M}(P, Q) \\ (DF) : (u(x), v(y)) \notin \mathcal{F}_{xy}^0 \\ (NC) : (x, y) \in \text{Supp}(\pi) \implies (u(x), v(y)) \in \mathcal{F}_{xy}. \end{cases}$$

- ▶ This is a **Nonlinear Complementarity Problem** (NCP) with much structure.
- ▶ Problem: existence of an equilibrium outcome? yes in the discrete case (\mathcal{X} and \mathcal{Y} finite): Kelso-Crawford, Alkan and Gale.

- ▶ Link with Galois connection, see Noeldeke and Samuelson (2015) and Larry's talk. Let

$$G_{xy}(v) = \alpha_{xy} + N(\gamma_{xy} - v).$$

One has $(u_x, v_y) \notin \mathcal{F}_{xy}^0$ if and only if $u_x \geq G_{xy}(v_y)$ which is equivalent to $v_y \geq G_{xy}^{-1}(u_x)$.

- ▶ By condition (DF) and (NC), we get that if (π, u, v) is the solution to an Equilibrium transportation problem

$$u(x) = \max_{y \in \mathcal{Y}} G_{xy}(v(y)) \text{ and } v(y) = \max_{x \in \mathcal{X}} G_{xy}^{-1}(u(x))$$

- ▶ OT is a special case (Φ -conjugacy, see Villani's book):

$$u(x) = \max_{y \in \mathcal{Y}} \{\Phi_{xy} - v(y)\} \text{ and } v(y) = \max_{x \in \mathcal{X}} \{\Phi_{xy} - u(x)\}$$

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- ▶ Assuming everything is smooth, and letting f_P and f_Q be the densities of P and Q we have under some conditions that the equilibrium transportation plan is given by $y = T(x)$, where mass balance yields

$$|\det DT(x)| = \frac{f_P(x)}{f_Q(T(x))}$$

and optimality yields

$$\partial_x G_{xT(x)}^{-1}(u(x)) + \partial_u G_{xT(x)}^{-1}(u(x)) \nabla u(x) = 0$$

which thus inverts into

$$T(x) = e(x, u(x), \nabla u(x)).$$

- ▶ Trudinger (2014) studies Monge-Ampere equations of the form

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- ▶ Galois connections and Monge-Ampère equations break the symmetry between x and y . Duality is an offspring of this broken symmetry.
- ▶ However, in contrast, the ET problem is symmetric in the role x and y play. Is there a way of restoring symmetry between x and y ?

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Section 3

DISTANCE FUNCTIONS

- ▶ [GKW] introduce distance function: for $(u, v) \in \mathbb{R}^2$, let

$$\Psi_{xy}(u, v) = \min \{t \in \mathbb{R} : (u - t, v - t) \in \mathcal{F}_{xy}\}$$

which is the distance along the diagonal between (u, v) and the frontier of \mathcal{F}_{xy} , with a minus sign if (u, v) is in the set.

- ▶ Economic interpretation: what is the quantity of utility that we can give or remove to x and y *in the same amount* such that they reach the efficient frontier?
- ▶ This object has nice properties:
 - ▶ $\Psi_{xy}(u, v) \leq 0$ iff $(u, v) \in \mathcal{F}_{xy}$
 - ▶ $\Psi_{xy}(u, v) < 0$ iff $(u, v) \in \mathcal{F}_{xy}^0$
 - ▶ $\Psi_{xy}(u + t, v + t) = \Psi_{xy}(u, v) + t$
- ▶ Note that in the case of OT,

$$\Psi_{xy}(u, v) = \frac{u + v - (\alpha_{xy} + \gamma_{xy})}{2}.$$

- ▶ (π, u, v) is an equilibrium outcome when

$$\left\{ \begin{array}{l} (PF) : \pi \in \mathcal{M}(P, Q) \\ (DF) : \Psi_{xy}(u(x), v(y)) \geq 0 \\ (NC) : (x, y) \in \text{Supp}(\pi) \implies \Psi_{xy}(u(x), v(y)) = 0. \end{array} \right.$$

- ▶ In the rest of the talk, I will argue that these objects can be useful in the study of the Equilibrium Transport problem.

- ▶ It is convenient to parameterize the frontier of \mathcal{F}_{xy} by a scalar parameter. To do this, consider a reparameterization by a 45° rotation: if u_x and v_y are such that $\Psi_{xy}(u_x, v_y) = 0$, let $w_{xy} = u_x - v_y$.
- ▶ More generally, the set of equations

$$\begin{cases} \Psi_{xy}(u_x, v_y) = 0 \\ u_x - v_y = w_{xy} \end{cases}$$

defines $u_x = \mathcal{U}_{xy}(w_{xy})$ and $v_y = \mathcal{V}_{xy}(w_{xy})$.

- ▶ One has a very simple expression of \mathcal{U}_{xy} and \mathcal{V}_{xy} from Ψ :

$$\begin{aligned} \mathcal{U}_{xy}(w_{xy}) &= -\Psi_{xy}(0, -w_{xy}) \\ \mathcal{V}_{xy}(w_{xy}) &= -\Psi_{xy}(w_{xy}, 0) \end{aligned}$$

and

$$G_{xy}(v) = \mathcal{U}_{xy} \circ \mathcal{V}_{xy}^{-1}(v) \quad \text{and} \quad G_{xy}^{-1}(u) = \mathcal{V}_{xy} \circ \mathcal{U}_{xy}^{-1}(u).$$

- ▶ Look at the individual rationality conditions:

$$\begin{cases} u_x \geq \mathcal{U}_{xy}(v_y) \\ \pi_{y|x} > 0 \implies u_x \geq \mathcal{U}_{xy}(v_y) \end{cases}$$

where $\pi_{y|x} = \pi_{xy}/p_x$ is the conditional distribution of Y given X under π .

- ▶ A regularized version of these is provided by the Gibbs distribution

$$\pi_{y|x} = \frac{\exp \mathcal{U}_{xy}(w_{xy}) / T}{\exp u_x / T}$$

where the max has been replaced by the smooth-max. Hence, letting $a_x = u_x - T \ln p_x$, we get

$$T \ln \pi_{xy} + a_x = \mathcal{U}_{xy}(w_{xy}).$$

- ▶ Similarly, individual rationality on the side of firms relaxes into

$$\pi_{x|y} = \frac{\exp \mathcal{V}_{xy}(w_{xy}) / T}{\exp v_y / T}$$

and thus, after letting $b_y = v_y - T \ln q_y$,

$$T \ln \pi_{xy} + b_y = \mathcal{V}_{xy}(w_{xy}).$$

- ▶ We get

$$\begin{cases} T \ln \pi_{xy} + a_x = \mathcal{U}_{xy}(w_{xy}) \\ T \ln \pi_{xy} + b_y = \mathcal{V}_{xy}(w_{xy}) \end{cases}$$

- ▶ Thus, applying Ψ term by term yields

$$\Psi_{xy}(T \ln \pi_{xy} + a_x, T \ln \pi_{xy} + b_y) = 0$$

hence

$$\pi_{xy} = \exp(-\Psi_{xy}(a_x, b_y) / T)$$

where a_x and b_y solve the system

$$\begin{cases} \sum_y \exp(-\Psi_{xy}(a_x, b_y) / T) = p_x \\ \sum_x \exp(-\Psi_{xy}(a_x, b_y) / T) = q_y \end{cases} .$$

- ▶ By subtraction, we have

$$a_x - b_y = w_{xy}.$$

- ▶ **Theorem:** when $T \rightarrow 0$, u and v tend to the solution of the ET problem.
- ▶ **Open question:** when there are multiple solutions at $T = 0$, which solution is selected?
- ▶ In the case of OT, it is the maximal entrop solution. Does this result still hold here?

- ▶ In OT, a lot of attention recently on the Monge-Kantorovich problem with entropic regularization

$$\max_{\pi \in \mathcal{M}(P, Q)} \int \Phi(x, y) d\pi(x, y) - 2T \int \log \pi(x, y) d\pi(x, y)$$

which is useful for computation (via Iterated Bregman projections) and econometrics (link w the logit model).

- ▶ In the OT case, this boils down to

$$\pi_{xy} = \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right)$$

where a and b are adjusted to meet the constraint $\pi \in \mathcal{M}(P, Q)$.

- ▶ In the ET case, no optimization formulation... but works just as well. Recall

$$\pi_{xy} = \exp(-\Psi_{xy}(a_x, b_y) / T).$$

POINT 2: COMPUTATION (CTD)

- ▶ Take temperature parameter $T > 0$ and look for π under the form

$$\pi_{xy} = \exp\left(-\frac{\Psi_{xy}(a_x, b_y)}{T}\right)$$

Note that when $T \rightarrow 0$, the limit of $\Psi_{xy}(a_x, b_y)$ is nonnegative, and the limit of $\pi_{xy} \Psi_{xy}(a_x, b_y)$ is zero.

- ▶ If $\pi_{xy} = \exp(-\Psi_{xy}(a_x, b_y)/T)$, condition $\pi \in \mathcal{M}(P, Q)$ boils down to set of nonlinear equations in (u, v)

$$\begin{cases} \sum_{y \in \mathcal{Y}} \exp\left(-\frac{\Psi_{xy}(a_x, b_y)}{T}\right) = p_x \\ \sum_{x \in \mathcal{X}} \exp\left(-\frac{\Psi_{xy}(a_x, b_y)}{T}\right) = q_y \end{cases}$$

which we call the *nonlinear Bernstein-Schrödinger* equation.

- ▶ In the optimal transportation case, this becomes the classical B-S equation

$$\begin{cases} \sum_{y \in \mathcal{Y}} \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) = p_x \\ \sum_{x \in \mathcal{X}} \exp\left(\frac{\Phi_{xy} - a_x - b_y}{2T}\right) = q_y \end{cases}$$

- ▶ Note that $F_x : a_x \rightarrow \sum_{y \in \mathcal{Y}} \exp\left(-\frac{\Psi_{xy}(a_x, b_y)}{T}\right)$ is a decreasing and continuous function. Mild conditions on Ψ therefore ensure the existence of a_x so that $F_x(a_x) = p_x$.
- ▶ Our algorithm [GKW] is thus a nonlinear Jacobi algorithm:
 - Make an initial guess of b_y^0
 - Determine the a_x^{k+1} to fit the p_x margins, based on the b_y^k
 - Update the b_y^{k+1} to fit the q_y margins, based on the a_x^{k+1} .
 - Repeat until b^{k+1} is close enough to b^k .
- ▶ One can prove that b_y^k decrease to fixed point. Convergence is very fast in practice.

- ▶ The distance function formalism allows to make a connection with Gale-Shapley's theory of stable marriages. Introduce

$$\Psi_{xy}(u, v) = \max(u - \alpha_{xy}, v - \gamma_{xy}).$$

- ▶ This interprets as: no matter what happens, x can get at most α_{xy} and y can get at most γ_{xy} .
- ▶ Assume $n_x = 1$ and $m_y = 1$ for all x and y . Then GH show that for any stable matching π in the Gale-Shapley sense, there exist u and v such that (π, u, v) is an equilibrium transport; and conversely, if (π, u, v) is an equilibrium transport, then π is stable in the Gale and Shapley sense.

- Back to the tax example [DGJK], assume that

$$N(w) = \min_{k=1 \dots K} \left\{ n^k + (1 - \theta^k) (w - w^k) \right\}$$

where $\theta^1 = 0 < \theta^2 < \dots < \theta^K$, and $n^0 = 0$,
 $n^{k+1} = n^k + (1 - \theta^k) (w^{k+1} - w^k)$.

- Recall $\mathcal{F}_{xy} = \{(u, v) : u - \alpha_{xy} \leq N(\gamma_{xy} - v)\}$. Because the tax schedule is progressive (N concave), \mathcal{F}_{xy} can be written as the intersection of the \mathcal{F}_{xy}^k , where

$$\mathcal{F}_{xy}^k = \left\{ (u, v) : u - \alpha_{xy} - n^k \leq (1 - \theta^k) (\gamma_{xy} - v - w^k) \right\}.$$

- Note that Ψ associated with the intersection of Ψ^k is equal to the $\max_k \Psi^k$. One has

$$\Psi^k(u, v) = \frac{u - \alpha_{xy} - n^k - (1 - \theta^k) (\gamma_{xy} - v - w^k)}{2 - \theta^k},$$

and as a result

$$\Psi(u, v) = \max_{k=1, \dots, K} \left\{ \frac{u - \alpha_{xy} - n^k - (1 - \theta^k) (\gamma_{xy} - v - w^k)}{2 - \theta^k} \right\}.$$

Thank you!

- ▶ More generally, the following operations on distance functions correspond to geometric operations on feasible sets:
 - ▶ $\max \{ \Psi^1, \Psi^2 \}$: intersection
 - ▶ $\min \{ \Psi^1, \Psi^2 \}$: union
 - ▶ $\Psi(u - \alpha, v - \gamma)$: translation
 - ▶ $T\Psi(u/T, v/T)$: homothety
 - ▶ $\lambda\Psi^1 + (1 - \lambda)\Psi^2$: interpolation
- ▶ These operations are exploited in the TraME project (<https://github.com/TraME-Project/>), a software for flexible computation of equilibrium transportation problems.