Optimal Martingale Transport

Alfred Galichon (NYU Econ+Math)

Based on joint works with Guillaume Carlier, Pierre Henry-Labordere, Nizar Touzi

November 2017

References

- Carlier and G (2012). "Exponential convergence for a convexifying equation." *Control, Optimization and Calculus of Variations*.
- G, Henry-Labordère and Touzi (2014). "A stochastic control approach to no-arbitrage bounds given marginals, with an application to Lookback options", *Annals of Applied Probability*.
- Tan and Touzi (2013). "Optimal transportation under controlled stochastic dynamics". *Annals of Probability*.

Section 1

FINANCIAL MOTIVATION

Consider a financial market with an asset X_t. Assume:

- X_t is a traded asset, so there is a process (σ_t) and a martingale measure under which $X = X^\sigma$, where

$$X_0^\sigma = x$$
$$dX_t^\sigma = \sigma_t dW_t$$

  where (W_t) is a Brownian motion under the martingale measure.
- There is a complete market of vanilla options at maturity $T = 1$, so the probability distribution Q of X_1 under the martingale measure is given

$$X_1 \sim Q.$$

- The volatility (σ_t) is uncertain, and there is no option market before maturity T that might lead to restrictions on (σ_t). Assume that we need to price an exotic option ξ whose underlying is the wole path $(X_t)_{t\in[0,1]}$. The lower bound on the price of ξ is given by

$$I(Q) = \inf_{(\sigma_t)} \left\{ \mathbb{E}[\xi] : X_0^\sigma = x,\ X_1^\sigma \sim Q \right\}.$$

Assume we want to price an option of maturity $T = 1$ on two underlyings X_1 and Y_1. The payoff of the option at date $T = 1$ is

$$\Phi\left(X_1, Y_1\right)$$

e.g. spread options $\Phi\left(X, Y\right) = \left(X - Y - k\right)^+$; cheapest to deliver $\Phi\left(X, Y\right) = \min\left(X, Y\right)$; etc.

Assume there is a perfectly liquid and complete market of single-name vanilla options on X_1 and Y_1, so that the risk neutral marginal probabilities P of X_1 and Q of Y_1 are known. Let $\mathcal{M}\left(P, Q\right)$ be the set of probabilities with these marginals.

The arbitrage bounds on the option price V are

$$\begin{gathered}\min_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi\left[\Phi\left(X_1, Y_1\right)\right] \\ \leq V \leq \\ -\min_{\pi \in \mathcal{M}(P,Q)} \mathbb{E}_\pi\left[-\Phi\left(X_1, Y_1\right)\right].\end{gathered}$$

which is a (classical) optimal transport problem.

Consider a European option of payoff $\varphi(X_1)$ at maturity 1, and assume there is a complete market for call and puts on X_1 at the same maturity. Let $P_0(k)$ and $C_0(k)$ the price at time 0 of these options, respectively. Then (Breeden-Litzenberger 1978)

$$\mathbb{E}[\varphi(X)] = \varphi(x) + \varphi'(x)(X_0 - x) + \int_0^x P_0(k)\varphi''(k)\,dk + \int_k^{+\infty} C_0(k)\varphi''(k)\,dk$$

hence:

- the vanilla market perfectly determines the risk-neutral probability of X at time 1
- replicating portfolios can be formed directly from vanilla options.

Consider

$$\min_{(X,Y)\sim\pi} \mathbb{E}_\pi \left[\Phi(X,Y)\right] \tag{P}$$
$$s.t. X \sim P,\ \ Y \sim Q.$$

Dual of this problem is

$$\max_{\varphi_0,\varphi_1} \mathbb{E}\left[\varphi_1(Y) - \varphi_0(X)\right] \tag{D}$$
$$s.t. \varphi_1(y) - \varphi_0(x) \leq \Phi(x,y).$$

Weak duality: easy. For $(X,Y) \sim \pi \in \mathcal{M}(P,Q)$, and (φ_0, φ_1) such that $\varphi_1(y) - \varphi_0(x) \leq \Phi(x,y)$, have

$$\varphi_1(Y) - \varphi_0(X) \leq \Phi(X,Y)$$

and taking expectations,

$$\mathbb{E}\varphi_1(Y) - \mathbb{E}\varphi_0(X) \leq \mathbb{E}\Phi(X,Y)$$

thus value of (D) weakly greater than value of (P). Converse relies on a separation theorem.

- The payoff $\varphi_1(y) - \varphi_0(x)$ is obtained from a subreplicating portfolio composed of vanilla single-name puts and calls. $\mathbb{E}\varphi_1(Y) - \mathbb{E}\varphi_0(X)$ its price.
- Monge-Kantorovich: Price of most expensive superreplicating portfolio=min price of option. Then φ_1 and φ_2 can be taken such that

$$\varphi_1(y) = \inf_x (\varphi_0(x) + \Phi(x,y))$$
$$\varphi_0(x) = \sup_y (\varphi_1(y) - \Phi(x,y))$$

(generalized convex duality).

- Now back to the original problem. Instead of pricing an option on underlying X and Y at time 1, the underlying is now X_1 and X_2, the price of the same asset at two dates forward.
- We still assume that the risk-neutral distributions can be implied from the option prices. The only difference with the previous setting is that we now have the restrictions implies by the fundamental law of asset pricing: under the risk-neutral distribution,

$$E\left[X_2|X_1\right] = X_1$$

- The upper bound on the option price is now

$$\min_{(X_1,X_2)\sim\pi} \mathbb{E}_\pi\left[\Phi\left(X_1,X_2\right)\right] \quad (1)$$

$$s.t. X_1 \sim P,\ X_2 \sim Q,\ E\left[X_2|X_1\right] = X_1$$

- The dual problem is

$$\max_{\varphi_0,\varphi_1} \mathbb{E}\left[\varphi_2\left(X_2\right) - \varphi_1\left(X_1\right)\right] \tag{2}$$
$$s.t. \varphi_2\left(x_2\right) - \varphi_1\left(x_1\right) - a\left(x_1\right)\left(x_2 - x_1\right) \leq \Phi\left(x_1, x_2\right).$$

- Interpretation: this is the optimal subreplicating portfolio made of vanilla calls and puts on X at maturities 1 and 2, plus rebalancing the quantity of the asset at period 1.

Section 2

Optimal semimartingale transport

- Consider the minimization of

$$A = \mathbb{E}\left[\int_0^1 L\left(t, X_t, \mu\left(t, X_t\right), \sigma\left(t, X_t\right)\right) dt\right]$$

over the processes X_t and drifts $\mu\left(t, x\right)$ as well as diffusion parameter σ such that

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t$$

and $X_0 \sim P$, $X_1 \sim Q$. For notational convenience, introduce $\Sigma = \sigma\sigma^*/2$, and assume $L\left(t, x, \mu, \Sigma\right)$.

- The value of the problem is given by

$$A = \min_{\mu, \Sigma, p} \int \int_0^1 L\left(t, x, \mu, \Sigma\right) p_t\left(x\right) dtdx \qquad (3)$$

subject to

$$p_0 = P,\ \ p_1 = Q$$

$$\partial_t p + \nabla.\left(p\mu\right) - \partial_{ij}^2\left(p\Sigma^{ij}\right) = 0$$

- The dual problem is

$$\max_{(\varphi_t)} \int \varphi_1 dQ - \int \varphi_0 dP \tag{4}$$
$$\partial_t \varphi + H\left(t, x, \nabla\varphi, D^2\varphi\right) = 0$$

where

$$H(t, x, p, M) = \max_{\mu,\Sigma} \left\{\mu.p + Tr(\Sigma M) - L(t, x, \mu, \Sigma)\right\}.$$

- Further

$$\varphi_t(y) = \inf \left\{ \begin{array}{c} \varphi_0(x) \\ +\mathbb{E}\left[\int_0^t L(t, X_t, \mu(t, X_t), \sigma(t, X_t))\, dt\right] \end{array} \right\}$$
$$s.t. X_1 = y$$
$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t.$$

The saddlefunction for this problem is

$$\begin{aligned}&\int\int_0^1 L\left(t,x,\mu,\Sigma\right)p_t\left(x\right)dtdx\\&+\int udp_0-\int udP-\int vdp_1+\int vdQ\\&+\int_0^1\int\varphi_t\left(\partial_t p+\nabla.\left(p\mu\right)-\partial^2_{ij}\left(p\Sigma^{ij}\right)\right)dxdt\end{aligned}$$

which is equal to

$$\begin{array}{c}\int\varphi_1 dQ-\int\varphi_0 dP\\-\int\int_0^1\left(\begin{array}{c}\partial_t\varphi_t+\mu.\nabla\varphi+Tr\left(\Sigma D^2\varphi\right)\\-L\left(t,x,\mu,\Sigma\right)\end{array}\right)p_t\left(x\right)dtdx\end{array}$$

The minimax formulation of the problem is

$$\max_{\varphi} \min_{p} \left\{ \begin{array}{c} \int \varphi_1 dQ - \int \varphi_0 dP \\ - \int \int_0^1 \left(\partial_t \varphi_t + H \left(t, x, \nabla \varphi, D^2 \varphi \right) \right) p_t \left(x \right) dt dx \end{array} \right\}$$

where

$$H \left(t, x, p, M \right) = \sup_{\mu, \Sigma} \left\{ \mu . p + Tr \left(\Sigma M \right) - L \left(t, x, \mu, \Sigma \right) \right\}$$

and one has the following expression for the dual problem

$$A = \max_{(\varphi_t)} \int \varphi_1 dQ - \int \varphi_0 dP$$

$$s.t.\ \partial_t \varphi + H \left(t, x, \nabla \varphi, D^2 \varphi \right) = 0.$$

See Tan and Touzi (2012).

Section 3

Skorohod embedding

Now assume that one wishes to constrain X_t to be a Markov martingale, i.e. $dX_t = \sigma(t, X_t)\, dW_t$. Then the value of the problem

$$A = \min_{\Sigma, p} \int \int_0^1 L(t, x, \Sigma)\, p_t(x)\, dt dx \tag{5}$$

subject to

$$p_0 = P,\ \ p_1 = Q$$

$$\partial_t p - \partial^2_{ij}\left(p \Sigma^{ij}\right) = 0$$

coincides with its dual formulation, that is

$$\max_{(\varphi_t)} \int \varphi_1 dQ - \int \varphi_0 dP \tag{6}$$

$$\partial_t \varphi + H\left(t, x, D^2 \varphi\right) = 0$$

where

$$H(t, x, M) = \max_{\Sigma} \left\{ Tr(\Sigma . M) - L(t, x, \Sigma) \right\}.$$

Assume

$$L(t,x,\Sigma) = 0 \text{ if } Tr(\Sigma) \leq 1$$
$$= +\infty \text{ otherwise.}$$

Then

$$H(t,x,M) = \max_{Tr(\Sigma)\leq 1} \{Tr(\Sigma.M)\}$$
$$= \max(Sp(M), 0)$$

the equation in (6) is

$$\partial_t \varphi + \max\left(Sp\left(D^2\varphi\right), 0\right) = 0$$

thus letting $\psi(t,x) = -\varphi(-t,x)$, the equation becomes

$$\partial_t \psi = \min\left(Sp\left(D^2\psi\right), 0\right) = 0$$

which is the convexification equation of L. Vese (1999), further studied in Carlier and Galichon (2012).

Section 4

Azéma-Yor from optimal transport

This section is based on G, Henry-Labordère and Touzi. Consider now X_t^σ a local martingale such that

$$X_0^\sigma = x$$
$$dX_t^\sigma = \sigma_t dW_t$$

and let $\bar{X}$ be its running maximum

$$\bar{X}_t^\sigma = \sup_{s\in[0,t]} X_s^\sigma .$$

- Consider the problem described in the introduction with $\xi = \bar{X}_1^\sigma$, that is

$$U(Q) = \sup_{\sigma_t} \mathbb{E}\left[\bar{X}_1^\sigma\right] \quad (7)$$
$$s.t.\ X_1 \sim Q$$

(we have thus assumed that $P = \delta_x$). The minmax formulation of this problem is

$$U(Q) = \inf_{\varphi} \sup_{\sigma_t} \mathbb{E}\left[\bar{X}_1^\sigma - \varphi\left(X_1^\sigma\right)\right] + \int \varphi dQ \quad (8)$$

and thus we are led to compute

$$\sup_{\sigma_t} \mathbb{E}\left[\bar{X}_1^\sigma - \varphi\left(X_1^\sigma\right)\right].$$

- Note that the problem above is equivalently given by

$$\sup_{\tau} \mathbb{E}\left[\bar{W}_\tau - \varphi\left(W_\tau\right)\right] \tag{9}$$

where the sup is taken over the stopping times τ.

- Note that formally

$$\varphi = \frac{\partial U}{\partial Q}$$

(this follows from the enveloppe theorem in (8)). In Machina's theory of "local utility", this means that φ is the local utility of U at Q.

Let g (resp. G) the pdf (resp. cdf) of Q, and let r be the upper bound of the support of Q. The following result holds:
Theorem. The solution to (7) is given by

$$U(Q) = \mathbb{E}_Q\left[b(Y)\right]$$

where b is Azéma-Yor's *barycenter function*

$$\begin{aligned} b(x) &= \mathbb{E}_Q\left[Y | Y \geq x\right] \text{ for } x < r \\ &= x \text{ for } x \geq r. \end{aligned}$$

Further, the optimal φ in (8) is given by

$$\varphi^*(x) = \int (x-y)^+ \frac{g(y)}{1-G(y)} dy$$

and the optimal stopping time in (9) is given by

$$\tau^* = \inf\left\{t > 0 : \bar{W}_t \geq b(W_t)\right\}.$$

Thus, the solution to this problem is given by the Azéma-Yor solution, and

$$\bar{W}_\tau \sim Q.$$

Step 1. φ may be taken convex.

Step 2. Take a given convex φ and introduce

$$u^{\varphi}(x,m) = \sup_{\tau \ s.t.} \mathbb{E}\left[\max\left(m, \bar{W}_{\tau}+x\right) - \varphi\left(W_{\tau}+x\right)\right].$$

Then u^{φ} has a PDE characterization as

$$\min\left\{u - m + \varphi(x), -\partial^2_{xx}u\right\} = 0 \text{ for } 0 < x < m$$

$$\partial_m u(m,m) = 0.$$

Step 3. Peskir (1998): let ψ be a maximal solution of ODE

$$\psi'(m) = \frac{1}{m - \psi(m)} \frac{1}{\varphi''(\psi(m))}$$

which lies strictly below the diagonal $\psi(m) < m$ (exists). Then

$$u^{\varphi}(x, m) = m - \varphi(x) + \int_{\psi(m)}^{\max(x, \psi(m))} (x - y) \varphi''(y) \, dy$$

so that the value of $u^{\varphi}(x, m) - m + \varphi(x)$ is

$$1\{\psi(m) < x\} \int_{\psi(m)}^{+\infty} (x - y)^{+} \varphi''(y) \, dy$$

and an optimal stopping time τ is given by

$$\tau = \inf\{t > 0 : \bar{W}_t \leq \psi(W_t)\}.$$

Step 4. Recall

$$U(Q) = \inf_{\varphi} \sup_{\sigma_t} \mathbb{E}\left[\bar{X}_1^{\sigma} - \varphi\left(X_1^{\sigma}\right)\right] + \int \varphi dQ$$
$$= \inf_{\substack{\varphi \text{ convex} \\ \varphi(x)=0}} u^{\varphi}(x,x) + \int \varphi dQ$$

thus

$$U(Q) = \inf_{\substack{\varphi \text{ convex} \\ \varphi(x)=0}} \left\{ \begin{array}{c} x + \int \varphi(y)\, g(y)\, dy \\ + \int_{\psi(x)}^{\max(x,\psi(x))} (x-y)\, \varphi''(y)\, dy \end{array} \right\}$$

hence

$$U(Q) = x + \inf_{\substack{\varphi \text{ convex} \\ \varphi(x)=0}} \left\{ \int c(y)\, \varphi''(y)\, dy \right\}$$

where $c(x) = \mathbb{E}_Q\left[(X-x)^+\right]$ is the European call price of strike x.

Step 5. By the change of variables $y = \psi(m)$, one has

$$
\begin{aligned}
U(Q) &= x + \inf_{\substack{\varphi \text{ convex} \\ \varphi(x)=0}} \left\{ \int c(\psi(m)) \, \varphi''(\psi(m)) \, \psi'(m) \, dm \right\} \\
&= x + \inf_{\psi(.)} \left\{ \int \frac{c(\psi(m))}{m - \psi(m)} dm \right\} \\
&= x + \int \inf_{\psi} \left\{ \frac{c(\psi)}{y - \psi} \right\} dy.
\end{aligned}
$$

Step 6. Let $x(y)$ minimizing

$$\inf_x \left\{ \frac{c(x)}{y-x} \right\}.$$

By first order conditions,

$$c'(x)(y-x) = c(x)$$
$$\Pr(Y \geq x)(y-x) = E\left[(Y-x)^+\right]$$

hence

$$y - x = \frac{E\left[(Y-x)^+\right]}{\Pr(Y \geq x)}$$
$$= b(x) - x$$

thus $y(x) = b(x)$.

Now,

$$\inf_x \left\{ \frac{c(x)}{y-x} \right\} = \Pr(Y \geq x(y))$$

thus

$$\begin{aligned} U(Q) &= x + \int \inf_\psi \left\{ \frac{c(\psi)}{y-\psi} \right\} dy = x + \int \Pr(Y \geq x(y))\, dy \\ &= x + \int (1 - G(x(y)))\, dy \end{aligned}$$

thus

$$U(Q) = \int b(y)\, g(y)\, dy.$$